STABILITY IN THE WHOLE OF INVARIANT SETS FOR NONAUTONOMOUS DIFFERENTIAL INCLUSION [1]

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In this work we continue to study (see [1]) different kinds of stability of positively invariant sets for a family of nonautonomous differential inclusions generated by a topological dynamical system. In particular, we discuss the conditions under which a positively invariant set is stable in the whole.

Let there be given a topological dynamical system (Σ, f^t) and a map $F : \Sigma\times\mathbb{R}^n \to \text{comp}(\mathbb{R}^n)$. Consider (for every $\sigma \in \Sigma$) the differential inclusion

$$\dot{x} \in F(f^t\sigma, x), \quad t \in \mathbb{R} \tag{1}$$

and the «convexified» differential inclusion

$$\dot{x} \in \text{co}F(f^t\sigma, x). \tag{2}$$

From now and on we assume that for every $\sigma \in \Sigma$ the function $F(f^t\sigma, x)$ is upper semicontinuous on x, bounded and uniformly continuous on $t \in \mathbb{R}$, and every solution of inclusion (1) is defined for all $t \geqslant 0$.

To every point $\omega = (\sigma, X) \in \Omega \doteq \Sigma\times\text{comp}(\mathbb{R}^n)$ we put into correspondence the section $S(t, \omega)$ of the integral funnel of inclusion (2) and a dynamical system (Ω, g^t), where $g^t\omega = (f^t\sigma, S(t, \omega))$. Next, for the given continuous function $\sigma \to M(\sigma) \in \text{comp}(\mathbb{R}^n)$ we construct the set $\mathfrak{M} \doteq \{\omega = (\sigma, X) \in \Omega : X \subset M(\sigma)\}$ and its r-neighborhood $\mathfrak{M}^r \doteq \{\omega = (\sigma, X) \in \Omega : X \subset M^r(\sigma)\}$, where $M^r(\sigma)$ is the r-neighborhood of the set $M(\sigma)$.

D e f i n i t i o n 1. The set $\mathfrak{M}$ is called: 1) *positively invariant* with respect to inclusion (1), if $g^t\mathfrak{M} \subset \mathfrak{M}$ for all $t \geqslant 0$; 2) *stably positively invariant* with respect to inclusion (1), if $\mathfrak{M}$ is positively invariant and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $g^t\mathfrak{M}^\delta \subset \mathfrak{M}^\varepsilon$ for every $t \geqslant 0$; 3) *stable in the whole* with respect to inclusion (1) if it is stably positively invariant and the deviation $d(S(t, \sigma, X), M(f^t\sigma))$ of the integral funnel $S(t, \sigma, X)$ from the set $M(f^t\sigma)$ tends to zero as $t \to \infty$ for every initial set $X \in \text{comp}(\mathbb{R}^n)$.

Denote $\mathfrak{N}^r \doteq \{\omega = (\sigma, x) \in \mathfrak{M}^r : \omega \notin \mathfrak{M}\}$ and let us have a continuous scalar function $\omega \to V(\omega)$, where $\omega = (\sigma, x) \in \mathfrak{M}^r$.

D e f i n i t i o n 2. Function V is said to be a *Lyapunov function* (with respect to the set $\mathfrak{M}$), if $V(\omega) = 0$ for all $\omega \in \partial\mathfrak{M}$ and $V(\omega) > 0$ for all $\omega \in \mathfrak{N}^r$; a Lyapunov function V is said to be *definitely positive* (with respect to the set $\mathfrak{M}$) if for every $\varepsilon \in (0, r)$ there exists $\delta > 0$ such that $V(\omega) \geqslant \delta$ for all $\omega \in \partial\mathfrak{M}^\varepsilon$.

We shall say that a function $\omega \to V(\omega)$ is *locally lipschitz* if for every $\sigma \in \Sigma$ and each $\vartheta > 0$ there exists l such that for any two points $(t_i, x_i) \in Q = \{(t, x) \in \mathbb{R}\times\mathbb{R}^n : |t| \leqslant \vartheta,\ x \in M^r(f^t\sigma)\}$, $i = 1, 2$, the inequality $|V(f^{t_1}\sigma, x_1) - V(f^{t_2}\sigma, x_2)| \leqslant l(|t_1 - t_2| + |x_1 - x_2|)$ takes place.

[1] The work in partially supported by RFBR (grant № 07-01-00305).

Let $r > 0$ and let a function $V : \mathfrak{M}^r \to \mathbb{R}$ be locally lipschitz. Then there exists the limit

$$V^o(\omega; q) \doteq \limsup_{(\vartheta,y,\delta)\to(0,x,+0)} \frac{V(f^{\delta\tau}(f^{\vartheta}\sigma), y+\delta h) - V(f^{\vartheta}\sigma, y)}{\delta}$$

which is called the *generalized derivative* of function V at the point $\omega = (\sigma, x)$ in the direction $q = (\tau, h) \in \mathbb{R} \times \mathbb{R}^n$ (or the Clarke derivative, [2]). If $q = (1, h)$, then $V^o_F(\omega) \doteq \max_{h\in F(\omega)} V^o(\omega; q)$ is said to be the *derivative of* V *with respect to inclusion* (1).

In the paper [1] it has been shown that the following statement is true.

T h e o r e m 1. *If there exists a locally lipschitz Lyapunov function* $V : \mathfrak{M}^r \to \mathbb{R}$ *such that* $V^o_F(\omega) \leqslant 0$ *for all* $\omega \in \mathfrak{N}^r$, *then the set* $\mathfrak{M}$ *is positively invariant with respect to inclusion* (1). *If, in addition,* V *is definitely positive, then the set* $\mathfrak{M}$ *is stably positively invariant with respect to inclusion* (1).

D e f i n i t i o n 3. (see [3]) A function $\omega = (\sigma, x) \to V(\omega) \in \mathbb{R}$ is called *infinite large* (with respect to the set $\mathfrak{M}$) if for every $R > 0$ there exists $r > 0$ such that for all $\omega \notin \mathfrak{M}^r$ the relation $V(\omega) \geqslant R$ holds.

Let $\alpha > 0$. Denote $\mathfrak{S}_\alpha \doteq \{\omega = (\sigma, x) \in \Omega : V(\omega) = \alpha\}$.

D e f i n i t i o n 4. We shall say that the set $\mathfrak{S}_\alpha$ *does not contain positive semitrajectories* of inclusion (2) if for every $\omega \in \mathfrak{S}_\alpha$ and each solution $\varphi(t, \omega)$ of inclusion (2) one can find $\tau > 0$ such that $V(g^\tau\omega) \neq \alpha$.

In other words, $\mathfrak{S}_\alpha$ does not contain positive semitrajectories of inclusion (2), if for every $\omega \in \mathfrak{S}_\alpha$ any dynamic $t \to g^t\omega = (f^t\sigma, \varphi(t, \omega))$, where $\varphi(t, \omega)$ is one of the solutions to inclusion (2), starting in $\mathfrak{S}_\alpha$ at $t = 0$ leaves $\mathfrak{S}_\alpha$ in a finite time.

T h e o r e m 2. *Let* Σ *be compact. If there exists a locally lipschitz definitely positive and infinite large function* $V : \Omega \to \mathbb{R}$ *such that* $V^o_F(\omega) \leqslant 0$ *for all* $\omega \notin \mathfrak{M}$, *and for every* $\alpha > 0$ *the set* $\mathfrak{S}_\alpha$ *does not contain positive semitrajectories of inclusion* (2), *then* $\mathfrak{M}$ *is stable in the whole with respect to inclusion* (1).

From this theorem there are also derived the statements on uniform (with respect to initial time moment) stability in the whole of the given set $(t, M(t)) \in \mathbb{R} \times \mathrm{comp}(\mathbb{R}^n)$ with respect to the ordinary differential inclusion $\dot{x} \in F(t, x)$ and controllable system $\dot{x} = f(t, x, u),\ u \in U$.

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Поступила в редакцию 25 апреля 2007 г.