GAUSS ALGEBRAS AND QUANTUM GROUPS [1]

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The category GA of Gauss algebras was introduced in [10] to study its objects by unified methods. In particular, GA contains (up to an equivalence) the category SLA of symmetrizable Lie algebras [3] and the category DJA of Drinfeld-Jimbo algebras [2]. The embedding SLA $\to$ GA is given by the universal functor $U : \mathfrak{g} \to U(\mathfrak{g})$. The language of GA is closely related to a study of famous quantum groups [1]. The aim of this paper is to focus on some aspects of this theory. A most of the text has a character of a review of relevant topics. Some new results are contained in Sections 10–12.

1 Main definitions

Let A be a unital associative algebra over a field F. We begin with the following preliminary definitions (GT), (CA).

(GT) A triple of subalgebras N_-, H, N_+ of the algebra A is called *Gauss triple* if they contain the unity of A and satisfy the following conditions:

(α) The algebra H normalizes $N_\pm : HN_\pm = N_\pm H$;

(β) The algebra A is equipped with $\mathbb{Z}$-grading A_n ($n \in \mathbb{Z}$) meaning that N_-, H, N_+ is generated (respectively) by homogeneous subsets e_-, e_0, e_+ consisting of elements x such that $\deg x < 0$, $\deg x = 0$, $\deg x > 0$; (γ) The algebra A admits a free (triangular) decomposition

$$A = N_- H N_+. \tag{1}$$

(CA) The algebra (1) is called *contragredient* (CA) if A is equipped with an (anti) involution $x \mapsto x'$ such that $h = h', A'_n = A_{-n}$ for any $h \in H$, $n \in \mathbb{Z}$.

Setting $e = e_+, e' = e_-$, we define the *canonical bilinear* form $\varphi : A \times A \to H$ by the rule

$$\varphi(x, y) = (x'y)_0, \tag{2}$$

where $x \mapsto x_0$ is defined as a projection $A \to H$ parallel to the subspace $T = e'A + Ae$.

Notice that $x_0 = x'_0$ for any $x \in A$. Hence the form (2) is symmetric. Moreover, its kernel $\ker \varphi$ contains the left ideal Ae. Hence φ is well defined on the quotient space

$$M = A/Ae. \tag{3}$$

It is clear that M is a cyclic A-module generated by the vector $1_+ = 1 + Ae$, such that $e1_+ = 0$. Using (α) (resp., (β)), we obtain that M inherits a structure of $A \times H$–bimodule (resp., $\mathbb{Z}$-grading) of A. Using (γ), we obtain an isomorphism of graded spaces

$$M = A1_+ = N_- H 1_+ \simeq B, \tag{4}$$

where $B = N_- H$. It is clear also that the grading A_n (resp., M_n) is orthogonal with respect to φ. The space M is called the *universal Verma module* of A.

[1] Supported by the Russian Foundation for Basic Research (grant No. 05-01-00001a).

The algebra A is called *nondegenerate* if the form φ is nondegenerate on M. Later we assume (for simplicity) $\operatorname{card} e < \infty$. Then we state the fundamental definition

(GA) The algebra (1) is called *Gauss algebra* (GA) if A is contragredient and nondegenerate.

It is easy to verify that the corresponding algebra H is commutative and without zero divisors. The algebra H is called *Cartan subalgebra* of A. The algebras $B_\pm = HN_\pm$ are called *Borel subalgebras* of A.

2 Main examples

We consider the following list of examples:

Weyl algebras $A = W_n$,

the algebras $A = U(\mathfrak{g})$, where $\mathfrak{g}$ is SLA,

the quantum envelopes (DJA) $A = U_q(\mathfrak{g})$, where $\mathfrak{g}$ is symmetrizable Kac-Moody algebra (SKMA),

superversions of these algebras,

an asymptotic version $A = D_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ (Kashiwara algebras),

some examples of translator algebras (Section 11),

the Yangians $Y = Y(\mathfrak{g})$ (Section 12).

Details are given below.

(1) The classical Weyl algebra $A = W_n$ is a unital algebra (over $\mathbb{C}$) generated by $2n$ elements e_i, f_i $(i = 1, ..., n)$ equipping with genetics $[e_i, e_j] = [f_i, f_j] = 0$ and

$$[e_i, f_j] = \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker symbol $(i, j = 1, ..., n)$. Setting $e = (e_i)$, $\deg e_i = 1$, $e_i' = f_i$, we obtain that A is GA with $M = \mathbb{C}[x]$, where $x = (x_1, ..., x_n)$. The action of A on M is given by the rule $e_i = \partial_i = \partial/\partial x_i$, $f_i = x_i$ $(i = 1, ..., n)$.

(2) We start from the known triangular decomposition of SLA: $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ [7]. Setting $A = U(\mathfrak{g})$ we obtain the corresponding decomposition (1), where $H = U(\mathfrak{h})$, $N_\pm = U(\mathfrak{n}_\pm)$. The algebra A is generated by elements $h \in \mathfrak{h}$ and Chevalley generators $e_i \in \mathfrak{n}_+$, $f_i \in \mathfrak{n}_-$ $(i \in I)$, where $\operatorname{card} I < \infty$. The obtained genetics is reduced to the known weight and Serre conditions [7] together with relations

$$[e_i, f_j] = \delta_{ij} h_i, \tag{6}$$

where $h_i \in \mathfrak{h}$ $(i \in I)$. Setting $e = (e_i)$, $\deg e_i = 1$, $e_i' = f_i$, we obtain that A is CA.

A study of the form φ is reduced to a study of a series of finite dimensional forms φ_n $(n \in \mathbb{Z}_+)$ associated to φ [4]. Calculating the principal terms of these (polynomial) forms, we obtain [7] that A is GA.

(3) The algebras $A = U_q(\mathfrak{g})$ (where $\mathfrak{g}$ is SKMA) are defined as quantum deformations of their classical versions $U(\mathfrak{g})$. It is known (see [7], for example) that A is GA.

(4) The algebras $A = D_q(\mathfrak{g})$ are introduced by M. Kashiwara [4] to a study of an asymptotic behaviour of $U_q(\mathfrak{g})$. In other words, $D_q(\mathfrak{g})$ can be defined as a "twisted differential algebra" of the quantum Serre algebra $S_q(\mathfrak{g})$ [8].

(5) A lot of superversions of the examples (1)–(4) can be easily defined. See [8] for example.

(6) A notion of translator algebras is discussed in Section 11. On occasion, these algebras are GA.

(7) The Yangian $A = Y(\mathfrak{g})$ was introduced by V. Drinfeld [1], [2] as a unique quantum deformation of the algebra $U(\mathfrak{g}[t])$, where $\mathfrak{g}$ is simple $(\dim \mathfrak{g} < \infty)$ and t is an independent variable. Sometimes $\mathfrak{g}$ can be reductive. If $\mathfrak{g}$ is simple, then A is GA (Section 12).

Later we consider some questions of theory of representations of GA. As a preliminary, we consider some special extensions of GA.

3 Projective extensions

Setting $e = (e_i)$, $i \in I$, we define e^n $(n \in \mathbb{Z}_+)$ as the set of monomials $e_{i_1}, ..., e_{i_n}$ $(e^0 = 1)$. Using (γ), we obtain that A is contained in the Cartesian product

$$A_{ext} = \prod_{n=0}^{\infty} Be^n, \tag{7}$$

where $B = B_-$. Elements $f \in A_{ext}$ can be considered as formal series with partial sums $f_n = \delta f_0 + ... + \delta f_n$, where $\delta f_n \in Be^n$. In this sence, A_{ext} coincides with the projective limit of spaces

$$A_n = A/Ae^{n+1}. \tag{8}$$

The algebra A is embedded in A_{ext} as the direct sum of subspaces Be^n (or the set of finite series in A_{ext}).

Proposition 1 [7]. *A_{ext} is an algebra with respect to the multiplication of formal series. Moreover, let A_{int} be the greatest $\mathbb{Z}$-graded subalgebra of A_{ext}. Then we have a chain of extensions*

$$A \subset A_{int} \subset A_{ext}. \tag{9}$$

Now, let $\mathcal{L}(e)$ be the category of A-modules X locally nilpotent with respect to $e : e^{n+1}x = 0$ for any $x \in X$ and any $n \geqslant n_0(x)$. Setting

$$fx = f_n x \quad \text{for} \quad n \geqslant n_0(x), \tag{10}$$

we define the action of A_{ext} on X. In other words, the module X is equipped with the structure of a A_{ext}-module.

Notice that $\mathcal{L}(e)$ is closed with respect to passing from a module to its submodule and its quotient module. If $X \neq 0$ (in $\mathcal{L}(e)$), then $X^e \neq 0$. Here the subspace

$$X^e = \ker e = \{x \in X : \ ex = 0\} \tag{11}$$

is called the *extremal subspace* of X (with respect to e).

In particular, the module M (Section 1) belongs to $\mathcal{L}(e)$. The action of the algebra A (and A_{ext}) on M is contained in the algebra

$$E(M) = \mathrm{End}_H M \tag{12}$$

(the commutant of the right action of H on M).

4 Perfect algebras

(PA) A Gauss algebra A is called *perfect* (PA) if H a field (an extension of F).

Theorem 1 [7] *If A is perfect, then the action of A on M determines an isomorphism of algebras*

$$A_{ext} \simeq E(M). \tag{13}$$

Respectively, we obtain an isomorphism of graded algebras

$$A_{int} \simeq E_{int}(M), \tag{14}$$

where $E_{int}(M)$ is the greatest $\mathbb{Z}$-graded subalgebra in $E(M)$.

Corollary 1 *The action of A (and A_{ext}) on M is exact. Moreover, M is a simple A-module.*

Corollary 2 *There exists a unique element $p \in A_{ext}$ satisfying equations*

$$ep = pe' = 0 \tag{15}$$

and normalyzing condition $p_{00} = 1$. Moreover, $p \in A_{int}$ and

$$p^2 = p = p', \ \deg p = 0, \tag{16}$$

with respect to the unique extension of $x \mapsto x'$ onto A_{int}.

Proposition 2 *The operator p projects any module X of category $\mathcal{L}(e)$ onto their extremal subspace X^e parallel to $e'X$. In particular, we have*

$$X = X^e \oplus e'X. \tag{17}$$

Example. Setting $A = W_1$, we obtain that any $f \in \operatorname{End}\mathbb{C}[x]$ can be written uniquely as a formal series:

$$f = \sum_{m,n=0}^{\infty} f_{mn} x^m \partial^n, \tag{18}$$

where $\partial = \partial/\partial x$, satisfying the finiteness condition (φ): for any n, only a finite set of coefficients $f_{mn} \in \mathbb{C}$ can differ from zero.

Remark. Assume that A has no zero divisors. Then the equation $p^2 = p$, i.e. $p(1-p) = 0$ has only trivial solutions $p = 0, 1$. Alternatively, the algebra A_{ext} admits a lot of projectors. The projector p given in Corollary 2 is called the *extremal projector* of algebra A.

Consider an application of the projector p in the theory of A–modules.

Proposition 3 *If A is PA, then the category $\mathcal{L}(e)$ is monoidal, with unique simple object M. In other words, any object X of $\mathcal{L}(e)$ has a form $X = nM$ (n is a cardinal number).*

Proof. For any module X in the category $\mathcal{L}(e)$, we set $X_0 = AX^e$. Assuming $X \neq X_0$, we obtain $(X/X_0)^e \neq 0$, i.e. there exists a vector $x \in X$ such that $x \notin Y$, $ex \subset Y$. Setting $y = px$, we obtain $y \subset Y$. Notice also that $x - px \in Y$. Hence $x \in Y$ (contradiction). As a result, we have $X = Y$.

Moreover, Y is the vector sum of a family of cyclic modules Ax_0, where $x_0 \in X^e$. Notice that the map $a1_+ \mapsto ax_0$ is correctly defined and determines an isomorphism $Ax_0 \simeq M$ for $x_0 \neq 0$ (because M is a simple A-module). Consequently, Y is the direct sum of submodules Ax_0, where x_0 runs a basis of X^e. Hence $X = Y = nM$ ($n = \dim X^e$). □

Examples:
1. $M = \mathbb{C}[x]$ for $A = W_n$.
2. $M = S_q(\mathfrak{g})$ for $A = D_q(\mathfrak{g})$.

5 Local extensions

(SA) A Gauss algebra A is called *standard* (SA) if A is invertible as H-bimodule, i.e. the left and right actions of an element $0 \neq h \in H$ in A is invertible. Hence A admits a structure of H'-bimodule, where $H' = \operatorname{Fract}H$ (the quotient field of H). Setting

$$A' = A \otimes_H H', \tag{19}$$

we obtain a natural structure of an algebra in (19). Namely, we set $af = a \otimes f$ in (19) and define fa by means of the normalyzing conditions (α), see Section 1, in $N_\pm$. The algebra (19) is called the *local extension* of A (with respect to H).

Notice that H' is the Cartan subalgebra of A'. Hence A' is perfect. Consequently, any SA has a perfect extension.

Notice that the extensions (7) and (19) are permutable. Respectively, we can define the *locally projective extension*

$$A'_{ext} = (A_{ext})' = (A')_{ext}.$$

In particular, any standard algebra A admits the extremal projector $p \in A'_{ext}$.

6 Cartan type algebras

(CTA) A Gauss algebra A is called the *Cartan type algebra* (CTA) if A admits a "fine" Q-grading A_μ ($\mu \in Q$), where Q is an ordered abelian subgroup in Aut$H$. The condition $x \in A_\mu$ means that

$$hx = xh_\mu \quad \text{forany} \quad h \in H, \tag{20}$$

where $\mu:\ h \mapsto h_\mu$ means the action of $\mu \in \mathrm{Aut}H$. Using the additive language for Q, we obtain a usual rule

$$A_\lambda A_\mu \subset A_{\lambda+\mu} \tag{21}$$

for any $\lambda, \mu \in Q$. The term "fine" means that $A_\mu \subset A_{n(\mu)}$, where the function $\mu \mapsto n(\mu)$ ($Q \to \mathbb{Z}$) is additive, $n(0) = 0$. Setting

$$Q_+ = \{\mu \in Q :\ \mu \geqslant 0\}, \tag{22}$$

we assume that the sets e_0, $e_\pm$ are homogeneous, $Q-\deg x = 0$ (resp., < 0, > 0) for $x \in e_0$ (resp., $x \in e_-$, $x \in e_+$). The Q-grading can be used (instead of $\mathbb{Z}$-grading) in the definition (GT), see Section 1.

For the case (CTA), the chain (9) can be completed as follows:

$$A \subset A_{fin} \subset A_{int} \subset A_{ext}, \tag{23}$$

where A_{fin} is the greatest Q-graded subalgebra in A_{ext}. It is also clear that any CTA is SA. Hence we have the corresponding chain in A':

$$A' \subset A'_{fin} \subset A'_{int} \subset A'_{ext}. \tag{24}$$

The indices in (23), (24) can be interpreted (respectively) as "fine", "integer", "extensive" (and also as "final", "internal", "external").

Notice that the universal Verma module M' for A' can be written as follows:

$$M' = A'/A'e = M \otimes_H H' \tag{25}$$

(a *localized Verma module* of A). Recall that the action of A' on M' determines an isomorphism of algebras $A'_{ext} \simeq E(M')$.

As usual, the algebra H consists of functions defined on some set Λ. Respectively, elements $f \in A'_{ext}$ can be considered as "rational functions" on Λ, except of a subset $\sigma(f)$ consisting of singularities of f.

7 Category $\mathcal{O}(e)$

Let A be CTA, $\mathcal{X}(H)$ be the set of characters of the algebra H. For any $\mu \in \mathcal{X}(H)$ and any H-module X we define the usual weight subspace

$$X_\mu\{x =\in X: \ hx = \mu(h)x, \text{ forany } h \in H\}. \tag{26}$$

Notice that the family (26) is linearly independent [7], and we have

$$A_\varepsilon X_\mu \subset X_{\varepsilon\mu}, \tag{27}$$

where $\varepsilon\mu(h) = \mu(h_\varepsilon)$ in terms of (20). A module X is called *H-diagonal* if it is graded by the subspaces (26), i.e.

$$X = \oplus_{\mu \in P} X_\mu, \tag{28}$$

for some subset $P \subset \mathcal{X}(H)$. Meaning (27), we assume that P is invariant with respect to Q ($QP \subset P$). It is essential that any submodule $Y \subset X$ inherits the grading (28), i.e.

$$Y = \oplus_{\mu \in P} Y_\mu, \tag{29}$$

where $Y_\mu Y \cap X_\mu$.

Let us fix a Q-submodule $P \subset \mathcal{X}(H)$. Let $\mathcal{O}(e)$ be the subcategory of $\mathcal{L}(e)$ consisting of P-graded (H-diagonal) A-modules (28). It is clear that $\mathcal{O}(e)$ is closed with respect to passing from a module to its submodules and quotient modules.

Example. For any $\lambda \in \mathcal{X}(H)$, we define the *Verma module*

$$M(\lambda) = M \otimes_B \mathbb{C}_\lambda, \tag{30}$$

where $B = B_+, \mathbb{C}_\lambda$ is an one-dimensional B-module defined by the character λ (extended trivially on N_+). Notice that

$$M(\lambda) = A/I_\lambda = M/J_\lambda, \tag{31}$$

where I_λ is a left ideal of A generated by the set e and by the elements $h_\lambda = h - \lambda(h)$, J_λ is a submodule of M generated by the elements h_λ ($h \in H$). Notice also that

$$M(\lambda) = A1_\lambda = N1_\lambda, \tag{32}$$

where $1_\lambda = 1 \otimes 1$ in (30), $1_\lambda = 1 + I_\lambda$ in the first part of (31).

Assume that Q acts effectively in P (i.e. the equality $\varepsilon\mu = \mu$ implies $\varepsilon = 1$ or $\mu = 0$). In that case, the relation $\lambda \geqslant \mu$ for $\lambda \in Q_+\mu$ determines an ordering in P. Using (32), we find that $M(\lambda)$ contains only weights $\mu \leqslant \lambda$, and $M(\lambda)_\lambda = \mathbb{C}1_\lambda$.

Moreover, there exists a greatest submodule $N(\lambda)$ of $M(\lambda)$ not containing the weight λ. Setting

$$V(\lambda) = M(\lambda)/N(\lambda), \tag{33}$$

we obtain a unique (up to an isomorphism) simple A-module with highest weight λ.

The theory of Verma modules (over CTA) is quite similar to that in classical case $A = U(\mathfrak{g})$. We concern some questions of this theory in Section 10.

8 Simplest cases

We consider the following four algebras: (1) $A = W_1$, (2) $A = U(\mathfrak{sl}_2)$, (3) $A = U_q(\mathfrak{sl}_2)$, (4) $A = D_q(\mathfrak{sl}_2)$, equipping (respectively) with Cartan subalgebra $H = \mathbb{C}, \mathbb{C}[h], \mathbb{C}(q), \mathbb{C}(q)$. All these algebras are equipped with two generators e, f and the following genetics:

(1) $[e, f] = 1$,
(2) $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$,
(3) $[e, f] = h_q,\ te = q^2et,\ tf = q^{-2}ft$ for $t = q^h,\ h_q = (q - q^{-1})^{-1}(t - t^{-1})$,
(4) $ef = q^2fe + 1$.

More exactly, in case (3) we can consider A as an algebra with generators e, f, t, t^{-1}. It is clear that all these algebras are CTA, and the algebras (1), (4) are PA. In all the four cases, the extremal projector p has the following form:

$$p = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!} f^n e^n \varphi_n^{-1}, \tag{34}$$

where $(n) = n$ for cases (1), (4), $(n) = [n]$ for cases (2), (3), and the factor $\varphi_n \in H$ has the form

$$\varphi_n = 1,\ \prod_{j=1}^{n}(h + j + z),\ \prod_{j=1}^{n}[h + j + z],\ 1 \tag{35}$$

in cases (1)–(4) (respectively). Here we use the symbol $[x]$ for $x = h+n$, where $n \in \mathbb{Z}$. Moreover, the projector (34) can be written as an infinite product

$$p = \prod_{n=1}^{\infty}(1 - \omega/a_n), \tag{36}$$

where $\omega = fe$, $a_n = n,\ n(h + n + 1),\ [n][h + n + 1],\ [n]q^{n-1}$ (respectively).

Recall that the category $\mathcal{L}(e)$ in cases (1), (4) is monoidal. The correspoding simple object M coincides with $\mathbb{C}[x]$, $S_q(\mathfrak{sl}_2)$ (respectively).

9 The case $A = U(\mathfrak{g})$

We consider the examples (1)–(7) of § 2. Notice that cases (1), (4) are simple (PA), cases (2), (3) are similar to each other (in view of quantization), and cases (5) correspond to (1)–(4). Cases (6), (7) will be considered later (Sections 11, 12). Hence in detail it suffices to consider case (2) only.

Let us fix the standard notations for $\mathfrak{g}$ [7]. In particular, let $\mathfrak{h}$ (resp., Δ_+) be a fixed Cartan subalgebra of $\mathfrak{g}$ (resp., a set of positive roots of $\mathfrak{g}$). For any $\lambda \in \mathfrak{h}^*$ we set $\lambda_\alpha = \lambda(h_\alpha)$, where $\alpha \in \Delta_+$. A vector $\rho \in \mathfrak{h}^*$ is defined by the condition $\rho_\alpha = 1$ for any simple $\alpha \in \Delta_+$.

Let p be the extremal projector of $\mathfrak{g}$ (i.e. of $A = U(\mathfrak{g})$). Notice that $p \in F(\mathfrak{g})_0$, where $F(\mathfrak{g}) = U(\mathfrak{g})'_{fin}$ (the localization of $U(\mathfrak{g})$ over the field $R(\mathfrak{h}) = \operatorname{Fract} U(\mathfrak{h})$). On the other hand, let z be (generalized) Casimir element of the algebra $\mathfrak{g}$ [7]. Notice that $z \in U(\mathfrak{g})_{fin}$ ($z \in U(\mathfrak{g})$ for $\dim\mathfrak{g} < \infty$) [7]. Then we have

$$z = z_0 + \omega, \tag{37}$$

where $x \mapsto x_0$ denotes the canonical projection onto H (Section 1), extended to the algebra $U(\mathfrak{g})_{fin}$. If is convenient to use the identification $H \simeq P(\mathfrak{h}^*)$ (the algebra of polynomial functions on $\mathfrak{h}^*$. Setting $z_\varepsilon(\lambda) = z_0(\lambda + \varepsilon)$, we obtain a family of affine functions $a_\varepsilon = z_\varepsilon - z_0$ ($\varepsilon \in \mathbb{Q}_+$):

$$a_\varepsilon(\lambda) = 2(\lambda + \rho, \varepsilon) + (\varepsilon, \varepsilon). \tag{38}$$

Theorem 2 [9], [7] *The extremal projector $p \in F(\mathfrak{g})$ can be written as infinite product*

$$p = \prod_{\varepsilon \neq 0} (1 - \omega / a_\varepsilon), \tag{39}$$

where $\omega = z - z_0$. The expression (39) is hold for any "constructive" $\mathfrak{g}$-module [9] (in particular, for M).

The expression (39) is called the *Lagrange form* of p. The following result means a "control on singularities" for p.

Theorem 3 [9], [7] *The function (38) defines an essential singularity of p only for $\varepsilon = j\alpha$ (quasiroot), where $\alpha \in \Delta_+$, $D \neq j \in \mathbb{Z}_+$.*

In particular, let $\dim \mathfrak{g} < \infty$. In that case, Theorem 4 follows from the known "normal form" of p [8]. Namely, for any normal ordering [8] $\Delta_+ = \{\alpha_1, ..., \alpha_m\}$, we have

$$p = p_{\alpha_1} ... p_{\alpha_m}, \tag{40}$$

where p_α is an analog of (34) corresponding to the root $\alpha \in \Delta_+$. It is essential that (40) does not depend on the choice of a normal ordering in Δ_+. An analog of (40) is valid also for affine Lie algebras [12].

Theorem 4 [9] *For any constructive $\mathfrak{g}$-module X and any $x \in X$, the left denomerator of the rational vector-function px has the form*

$$\pi_x = \prod_{\alpha \in \Delta_+} \prod_{j=1}^{m_\alpha} (h_\alpha + \rho_\alpha + j), \tag{41}$$

where $m_\alpha = \max\{n : e_\alpha^n x \neq 0\}$.

The latter result is nontrivial as a control on singularities for the function px. For example, let $e_\alpha x = 0$ for a fixed $\alpha \in \Delta_+$. Then the factor p_α can be omitted in px, independently on its position in (40).

10 Verma modules

Let A be CTA. We select briefly a way of using the projector p in the theory of A-modules in the category $\mathcal{O} = \mathcal{O}(e)$.

Proposition 4 *The singular set $\sigma(p)$ (Section 6) coincides with the following subsets of $\lambda \in \mathfrak{h}^*$:*
(α) there exists a nontrivial homomorphism $M(\lambda) \to M(\lambda + \varepsilon)$, for some $\varepsilon \in \mathbb{Q}_+$,
(β) the module $V(\lambda)$ coincides with a factor of $M(\lambda + \varepsilon)$ for some $\varepsilon \in \mathbb{Q}_+$,
(γ) the module $M(\lambda)$ is not projective.

In particular, let the algebra $N = N_-$ is without zero divisors. Then the action of N in M is free. In particular, any nontrivial homomorphism $M(\lambda) \to M(\lambda + \varepsilon$ is an injection.

Let X be an object of the category $\mathcal{O}$. We define $P(X)$ as a set of $\mu \in P$ such that $x_\mu \neq 0$. A vector $x \in X_\mu$ is called *primitive* if $x \notin Y$, $ex \subset Y$ for some submodule Y of X. In that case the correspoding weight μ is called primitive. A subset of $P(X)$ consisting of primitive weights is denoted $P_{prim}(X)$.

A module X (of category $\mathcal{O}$) is called *regular* if any primitive vector $x \in X_\mu$ is extremal ($ex = 0$) or the corresponding weight μ is regular in the following sence: $\mu \notin \sigma(p)$. The subcategory of $\mathcal{O}$ consisting of regular modules is denoted by $\mathcal{O}_{reg}$.

Proposition 5 *Any module X of the category $\mathcal{O}$ is generated by its primitive vectors. Any module X of the category $\mathcal{O}_{reg}$ is generated by its extremal vectors. In the latter case, we have*

$$X = AX^e. \tag{42}$$

Theorem 5 *The category $\mathcal{O}_{reg}$ is semisimple, with simple objects $V(\lambda)$.*

The proof is similar to that of Proposition 3. In particular, for $A = U(\mathfrak{g})$, the category $\mathcal{O}_{reg}$ contains the subcategory $\mathcal{O}_{int}$ of "integrable" A-modules [3]. In the case $\dim\mathfrak{g} < \infty$, the category $\mathcal{O}_{int}$ coincides with the category $\mathfrak{F}$ of finite dimensional $\mathfrak{g}$-modules. In that case, theorem 6 coincides with a classical theorem by H. Weyl [7].

11 Hypersymmetry

We draw the attention to a special way of constructing of GA. A pair of unital algebras (A, B) is called *admissible* if $1 \in B \subset A$, where B is GA and 1 is the unity of A. For any A-module X, we set $X^e = \ker e$ (with respect to B). The space

$$T = \{a \in A : ea \subset Ae\} \tag{43}$$

coincides with $\operatorname{Norm} Ae$ (= the greatest subalgebra of A containing Ae as a two-sided ideal). It is clear that $TX^e \subset X^e$ (for any A-module X). Setting

$$S = T/Ae, \tag{44}$$

we define an action of S in X^e (via $eX^e = 0$). Hence we have $SX^e \subset X^e$. The algebra (44) is called the *hypersymmetry algebra* of the space X^e.

Using (43), we obtain that S is a part of M. Passing to the rational hull $M' = A'/A'e$ (Secton 6), we obtain the algebra

$$Z = S' = pM', \tag{45}$$

where p is the extremal projector of B ($p \in B'_{ext}$). The following example is crucial (in the theory of Lie algebras).

The pair $(\mathfrak{g}, \mathfrak{k})$ of finite dimensional Lie algebras is called *reductive* if the algebra $\mathfrak{k}$ is reductively embedded in $\mathfrak{g}$ (i.e. $\mathfrak{g}$ is a reductive $(\operatorname{ad}\mathfrak{k})$-module). In particular, $\mathfrak{k}$ is reductive and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is a complementary $(\operatorname{ad}\mathfrak{k})$-module.

It is clear that the pair $A = U(\mathfrak{g})$, $B = U(\mathfrak{k})$ is admissible. In that case, the localization (45) is given over the field $R(\mathfrak{h}) = \operatorname{Fract} U(\mathfrak{h})$, where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{k}$.

Theorem 6 [8] *Let a_i $(i = 1, ..., n)$ be a weight basis of the* $(\operatorname{ad}\mathfrak{k})$*-module* $\mathfrak{p}$. *Then the algebra $Z = Z(\mathfrak{g}, \mathfrak{k})$ defined in (45) is generated (over $R(\mathfrak{h})$) by the elements $z_i = pa_i$ $(i = 1, ..., n)$ equipping with a quadratic genetics (over $R(\mathfrak{h})$).*

Example. Set $AZ_n = Z(\mathfrak{g}_n, \mathfrak{g}_{n-1})$, where $\mathfrak{g}_n = \mathfrak{gl}_n$ (over $\mathbb{C}$). The elements $e_i = e_{in}$, $e_{-i} = e_{ni}$ $(i = 1, ..., n-1)$ and $e_0 = e_{nn}$ form a weight basis of $\mathfrak{p}$. Respectively, AZ_n is generated by the elements $z_i = pe_i$ $(i = 0, \pm 1, ..., \pm(n-1))$. The corresponding genetics has the following form:

$$z_i z_j = \varepsilon_{ij} z_j z_i, \quad \text{for} \quad i + j \neq 0, \tag{46}$$

$$z_i z_{-i} = \sum_{j=1}^{n-1} \alpha_{ij} z_{-j} z_j + \gamma_i = 0 \tag{47}$$

with coefficients in $R(\mathfrak{h})$ [8]. In that case, the algebra AZ_n is GA.

12 Yangians

The Yangian $Y(\mathfrak{g})$ was introduced by V. Drinfeld [1], [2] as an important example of a quantum group. Here $\mathfrak{g}$ is a complex simple Lie algebra. The algebra $A = Y(\mathfrak{g})$ is a unital algebra over $\mathbb{C}$. Moreover, V. Drinfeld described a genetics of $Y(\mathfrak{g})$, in terms of special elements H_{iz}, $X^{\pm}_{iz}$ [1]. Actually, the algebra $A = Y(\mathfrak{g})$ has a triangular decomposition (1), in terms of these generators. Moreover, A is CA.

It is not difficult to verify (by the analogy with $U(\mathfrak{g})$) that the canonical form of $Y(\mathfrak{g})$ is nondegenerate (on M). In other words, we obtain the following

Theorem 7 *The Yangian $Y(\mathfrak{g})$ ($\mathfrak{g}$ is simple) is GA.*

The theory of finite dimensional representations of $Y(\mathfrak{g})$ [10] can be easily embedded in the general theory of representations of GA. On the other hand, the definition of $Y(\mathfrak{g})$ can be sometimes extended to the case when $\mathfrak{g}$ is a complex reductive Lie algebra ($\mathfrak{g} = \mathfrak{gl}_n$, for example).

The Yangian $Y_n = Y(\mathfrak{gl}_n)$ is a unital algebra (over $\mathbb{C}$) generated (over $\mathbb{C}$) by the set of elements

$$t_{ij}(u) = \sum_{s=0}^{\infty} t_{ij}^{(s)} u^{-s}, \; t_{ij}^{(0)} = \delta_{ij}, \tag{48}$$

where u is an independent variable (in fact Y_n is generated by the coefficients in (48)). The correspoding genetics of Y_n has the following form:

$$[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u). \tag{49}$$

Example ($n = 2$). In the case, the matrix $t(u) = (t_{ij}(u))$ can be rewritten as follows:

$$t(u) = \begin{pmatrix} \alpha(u) & \beta(u) \\ \gamma(u) & \delta(u) \end{pmatrix}, \tag{50}$$

where

$$x(u) = \sum_{i=0}^{\infty} x_i u^{-i}$$

for $x = \alpha, \beta, \gamma, \delta$. Setting $(xy)_{ij} = [x_i, y_j]$, we can write the genetics (49) in the following form:

$$(xy)_{i+1,j} - (xy)_{i,j+1} = (\overline{xy})_{ij}, \tag{51}$$

where the symbol $\overline{xy}$ is defined in terms of permutation of rows $x \mapsto \overline{x}$ in (50). Namely, $\overline{xy} = xy$ (resp., $\overline{x}\,\overline{y}$) if the elements x, y belong to a common row (resp., to distinct rows) in the matrix (50).

Using the genetics (51), we obtain that $Y = Y_2$ has triangular decomposition (1), where H (resp., N_-, N_+) is generated by the elements $\alpha(u), \delta(u)$ (resp., $\gamma(u), \beta(u)$). Moreover, $H = AD = DA$, where A (resp., D) is generated by the elements $\alpha(u)$ (resp., $\delta(u)$). The algebra H is not commutative. Hence this decomposition does not imply the structure of GA.

However, it is easy to verify that the algebra Y coincides with a projective limit of Gauss algebras $\pi(Y)$, where π is a finite dimensional representation of Y.

In particular, the general theory of representations of GA (in the category $\mathcal{O}(e)$) can be used to study finite dimensional representations of Y.

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