MSC 43A85, 32D15

Analytic continuation of spherical functions for the hyperboloid of one sheet [1]

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For the hyperboloid of one sheet in $\mathbb{R}^3$, we construct four complex hulls and continue analytically to them spherical functions of different series (continuous, holomorphic and antiholomorphic)

Keywords: hyperboloid of one sheet, complex hulls, spherical functions

In this paper we study analytic continuation of spherical functions on the hyperboloid of one sheet $\mathcal{X}$ in $\mathbb{R}^3$ on complex hulls of the hyperboloid.

The quasiregular representation of the group $G = \mathrm{SO}_0(1,2)$ decomposes in irreducible unitary representations of the continuous series (with multiplicity 2) and the holomorphic and anti-holomorphic discrete series (with multiplicity 1). This decomposition is equivalent to the decomposition of the delta function on $\mathcal{X}$ on spherical functions of these series.

It was known [2], [3] that spherical functions of *discrete* series can be continued analytically on some complex manifolds (complex hulls of the hyperboloid $\mathcal{X}$). But a similar question for spherical functions of the *continuous* series was not studied.

In this paper we construct 4 complex hulls $\mathcal{Y}^+$, $\mathcal{Y}^-$, Ω^+ and Ω^- of the hyperboloid $\mathcal{X}$. The spherical functions of two discrete series can be continued analytically on two first manifolds, each series needs its own hull. But for the spherical functions of the continuous series, the situation is more complicated and more interesting: the spherical functions of the continuous series need both manifolds Ω^+ and Ω^-, each spherical function is half the sum of the limit values from Ω^+ and Ω^-.

§ 1. Complex hulls

Introduce in the space $\mathbb{R}^3$ the bilinear form

$$[x,\, y] = -x_1y_1 + x_2y_2 + x_3y_3, \tag{1.1}$$

[1] Supported by the Russian Foundation for Basic Research (RFBR): grant 09-01-00325-a, Sci. Progr. "Development of Scientific Potential of Higher School": project 1.1.2/9191, Fed. Object Progr. 14.740.11.0349 and Templan 1.5.07

The hyperboloid of one sheet $\mathcal{X}$ is defined by the equation $[x,\, x] = 1$. Let $\mathcal{X}^{\mathbb{C}}$ be the complexification of $\mathcal{X}$. It is obtained as follows. Let us extend the bilinear form $[x,\, y]$ to the space $\mathbb{C}^3$ by the same formula (1.1). Then $\mathcal{X}^{\mathbb{C}}$ is the set of points x in $\mathbb{C}^3$ satisfying the equation $[x,\, x] = 1$. Its complex dimension is equal to 2.

We consider that the group $G = \mathrm{SO}_0(1, 2)$ acts linearly on $\mathbb{R}^3$ from the right: $x \mapsto xg$. In accordance with it we write vectors in the row form. On the hyperboloid $\mathcal{X}$, the group G acts transitively. The group G also acts on $\mathcal{X}^{\mathbb{C}}$: $x \mapsto xg$, but of course not transitively.

In this section we determine some complex manifolds in $\mathcal{X}^{\mathbb{C}}$ of complex dimension 2, invariant with respect to G. They are maximal in some sense, the group G acts on them simply transitively, so that the G-orbits are diffeomorphic to G and have real dimension 3. The hyperboloid $\mathcal{X}$ is contained in the boundary of each of these manifolds. We call them the "complex hulls" of the hyperboloid $\mathcal{X}$.

We need the group $\mathrm{SL}(2, \mathbb{C})$ and its subgroup $\mathrm{SU}(1, 1)$. They consist respectively of matrices:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad g_1 = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

with the unit determinant. The group $\mathrm{SL}(2, \mathbb{C})$ acts on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the Riemann sphere) linear fractionally:

$$z \mapsto z \cdot g = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

This action is transitive. But the subgroup $\mathrm{SU}(1, 1)$ has three orbits on $\overline{\mathbb{C}}$: the open disk $D : z\overline{z} < 1$, its exterior $D' : z\overline{z} > 1$, and the circle $S : z\overline{z} = 1$.

Denote the group $\mathrm{SU}(1, 1)$ by G_1, then the group $\mathrm{SL}(2, \mathbb{C})$ is its complexification $G_1^{\mathbb{C}}$.

Let us identify the space $\mathbb{R}^3$ with the space of matrices

$$x = \begin{pmatrix} ix_1 & x_2 + ix_3 \\ x_2 - ix_3 & -ix_1 \end{pmatrix}.$$

The action $x \mapsto g^{-1}xg$ of the group G_1 on these matrices x is the action $x \mapsto xg$ of the group G on vectors $x \in \mathbb{R}^3$. It gives an homomorphism of G_1 on G.

Introduce on $\mathcal{X}$ horospherical coordinates u, v: $(u, v) \in S \times S$, $u \neq v$, by

$$x = \left(\frac{u + v}{i(u - v)}, \frac{1 - uv}{u - v}, \frac{1 + uv}{i(u - v)} \right).$$

The inverse map $x \mapsto (u, v)$ is given by

$$u = \frac{x_3 + ix_2}{x_1 + i}, \quad v = \frac{x_3 + ix_2}{x_1 - i}. \tag{1.2}$$

It embeds the hyperboloid $\mathcal{X}$ into the torus $S \times S$, the image is the torus without the diagonal $\{u = v\}$, the diagonal is a boundary of the hyperboloid.

When a point $x \in \mathcal{X}$ is transformed by $g \in G$, its coordinates (u, v) are transformed by a fractional linear transformation (the same for u and v): $u \mapsto u \cdot g_1$,

$v \mapsto v \cdot g_1$, where g_1 is an element in the group G_1 which goes to $g \in G$ under the homomorphism $G_1 \to G$ mentioned above.

Similarly we introduce horospherical coordinates z, w on $\mathcal{X}^{\mathbb{C}}$: a point $x \in \mathcal{X}^{\mathbb{C}}$ is

$$x = \left(\frac{z+w}{i(z-w)}, \frac{1-zw}{z-w}, \frac{1+zw}{i(z-w)} \right), \tag{1.3}$$

the variables z, w run over the extended complex plane $\overline{\mathbb{C}}$, with the condition $z \neq w$. The inverse map is given by

$$z = \frac{x_3 + ix_2}{x_1 + i}, \quad w = \frac{x_3 + ix_2}{x_1 - i}. \tag{1.4}$$

These formulae, defined originally for $x_1 \neq \pm i$, are extended by continuity to all $x \in \mathcal{X}^{\mathbb{C}}$. Namely, the points $x = (i, i\lambda, \lambda)$, $x = (-i, i\lambda, \lambda)$, $\lambda \neq 0$, have horospherical coordinates $(0, i/\lambda)$, $(-i/\lambda, 0)$, respectively, and the points $x = (i, -i\lambda, \lambda)$, $x = (-i, -i\lambda, \lambda)$ have horospherical coordinates $(-i\lambda, \infty)$ and $(\infty, i\lambda)$, respectively.

Thus, formulae (1.4) give an embedding $X^{\mathbb{C}} \to \overline{\mathbb{C}} \times \overline{\mathbb{C}}$, the image is $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ without the diagonal.

There are the following relations. Let the points x and y in $\mathcal{X}^{\mathbb{C}}$ have horospherical coordinates (z, w) and (λ, μ) respectively. Then

$$[x, y] - 1 = -\frac{2(z-\lambda)(w-\mu)}{(z-w)(\lambda-\mu)}, \tag{1.5}$$

$$[x, y] + 1 = -\frac{2(z-\mu)(w-\lambda)}{(z-w)(\lambda-\mu)}. \tag{1.6}$$

If a point x in $\mathcal{X}^{\mathbb{C}}$ has horospherical coordinates (z, w), then the point $\overline{x}$ (it belongs to $\mathcal{X}^{\mathbb{C}}$ too) has horospherical coordinates $(1/\overline{z},\ 1/\overline{w})$. Together with (1.5), (1/6) this gives

$$[x, \overline{x}] - 1 = 2\,\frac{(1-z\overline{z})(1-w\overline{w})}{|z-w|^2}, \tag{1.7}$$

$$[x, \overline{x}] + 1 = 2\left|\frac{1-z\overline{w}}{z-w}\right|^2. \tag{1.8}$$

Moreover, for the imaginary parts we have

$$\operatorname{Im} x_1 = -\frac{z\overline{z} - w\overline{w}}{|z-w|^2}, \tag{1.9}$$

$$\operatorname{Im} \frac{x_3}{x_2} = -\frac{1 - z\overline{z} \cdot w\overline{w}}{|1-zw|^2}. \tag{1.10}$$

In the direct product $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ let us consider 4 complex manifolds:

$$D \times D,\ D' \times D',\ D \times D',\ D' \times D. \tag{1.11}$$

The torus $S \times S$ is contained in the boundary of each of them. The group G_1 acts on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ diagonally: $(z, w) \mapsto (z \cdot g, w \cdot g)$. It preserves all these manifolds (1.11). But this action is not transitive. This can be seen already when we compare dimensions: dimesion of G_1 is less than dimension of each manifold ($3 < 4$). Further, the group G_1 preserves $[x, \overline{x}]$, therefore, by (1.8), it preserves, for instance,

$$J = \frac{[x, x+1}{2} = \left| \frac{1 - z\overline{w}}{z - w} \right|^2 ,$$

so that any G_1-orbit lies on the level surface $J = const$.

Lemma 1.1 *The following pairs are representatives of G_1-orbits:*

$$\begin{aligned} &(-i\mu, i\mu),\ 0 \leqslant \mu < 1,\ \text{for } D \times D, \\ &(-i\mu, i\mu),\ 1 < \mu \leqslant \infty,\ \text{for } D' \times D', \\ &(-i\mu, i\mu^{-1}),\ 0 \leqslant \mu < 1,\ \text{for } D \times D', \\ &(-i\mu, i\mu^{-1}),\ 1 < \mu \leqslant \infty,\ \text{for } D' \times D. \end{aligned}$$

Proof. Let us consider $D \times D$. Since G_1 acts transitively on D, we can move the first element of a pair in $D \times D$ to zero. We obtain a pair $(0, \zeta)$, $\zeta \in D$. Now we can act on this pair by the centralizer of 0, i.e the diagonal group K_1 of G_1. It consists of matrices g_1 with $a = \mathrm{e}^{i\alpha}, b = 0$. They act by rotations with angle 2α around zero. So we can move ζ to ir, $0 \leqslant r < 1$. Thus, any pair in $D \times D$ can be moved to a pair $(0, ir)$, $0 \leqslant r < 1$. This pair can be transferred to a pair $(-i\mu,\ i\mu)$ in the lemma by means of a matrix

$$g_1 = \frac{1}{\sqrt{1-\mu^2}} \begin{pmatrix} 1 & i\mu \\ -i\mu & 1 \end{pmatrix}, \quad r = \frac{2\mu}{\mu^2+1}.$$

Similarly we consider the other 3 cases. □

For all μ satisfying the strong inequalities in Lemma 1.1, i. e. $0 < \mu < 1$ or $1 < \mu < \infty$, the stabilizer of the pair indicated in the lemma is the center $\{\pm E\}$ of the group G_1, so that the G_1-orbits of these pairs are diffeomorphic to the group $G \simeq G_1/\{\pm E\}$ and has dimension three.

For $\mu = 0$ or $\mu = \infty$ the stabilizer of the pairs is the subgroup K_1in G_1 consisting of diagonal matrices, so that the corresponding G_1-orbits are diffeomorphic to the Lobachevsky plane $\mathcal{L} = G_1/K_1$ and have dimension two. For $D \times D$ and $D' \times D'$ these two-dimensional orbits are the diagonals $\{z = w\}$, and for $D \times D'$ and $D' \times D$ they are the manifolds $\{z\overline{w} = 1\}$. Indeed, the matrix g_1 carries the pairs $(0, 0)$, $(\infty,\ \infty)$, $(0,\ \infty)$, $(\infty,\ 0)$ to the pairs $(z,\ z)$, $(w,\ w)$, $(z,\ w)$, $(w,\ z)$ respectively, where $z = \overline{b}/\overline{a}$, $w = a/b$, so that $z\overline{w} = 1$.

Let us delete these two-dimensional orbits from the manifolds (1.11) and denote the remainig manifolds by the same symbols with index 0, for example, $(D \times D)_0$ etc. For these manifolds, the representatives of the G_1-orbits are the pairs indicated in Lemma 1.1 with μ satisfying the inequalities $0 < \mu < 1$ or $1 < \mu < \infty$.

Let us go from $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ to $\mathcal{X}^{\mathbb{C}}$ by (1.3) and (1.4). The images of $(D \times D)_0$, $(D' \times D')_0$, $(D \times D')_0$, $(D' \times D)_0$ will be denoted by $\mathcal{Y}^+$, $\mathcal{Y}^-$, Ω^+, Ω^- respectively.

By (1.7) – (1.10) we get the folowing description of these sets (recall that they all lie in $\mathcal{X}^{\mathbb{C}} : [x,\, x] = 1$):

$$\begin{array}{lll} \mathcal{Y}^+ & : & [x,\, \overline{x}] > 1, \quad \operatorname{Im} \dfrac{x_3}{x_2} < 0, \\ \mathcal{Y}^- & : & [x,\, \overline{x}] > 1, \quad \operatorname{Im} \dfrac{x_3}{x_2} > 0, \\ \Omega^+ & : & -1 < [x,\, \overline{x}] < 1, \quad \operatorname{Im} x_1 > 0, \\ \Omega^- & : & -1 < [x,\, \overline{x}] < 1, \quad \operatorname{Im} x_1 < 0, \end{array}$$

The pairs $(-i\mu,\, i\mu)$ and $(-i\mu,\, i\mu^{-1})$ go to the points in $\mathcal{X}^{\mathbb{C}}$ lying on the curves

$$y_t = (0,\, i \sinh t,\, \cosh t), \quad \omega_t = (i \sin t,\, 0, \cos t), \tag{1.12}$$

where $\mu = \mathrm{e}^{-t}$, $\mu = \tan(\pi/4 - t/2)$, respectively. Representatives of the G_1-orbits are the points:

$$\begin{array}{rl} y_t : & t > 0, \quad \text{for } \mathcal{Y}^+, \\ y_t : & t < 0, \quad \text{for } \mathcal{Y}^-, \\ \omega_t : & 0 < t < \pi/2, \quad \text{for } \Omega^+, \\ \omega_t : & -\pi/2 < t < 0, \quad \text{for } \Omega^-. \end{array}$$

The Lie algebra $\mathfrak{g}$ of the group $G = \mathrm{SO}_0(1,2)$ consists of matrices

$$X = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & -\xi_0 \\ \xi_2 & \xi_0 & 0 \end{pmatrix}.$$

Let us take the corresponding basis in $\mathfrak{g}$:

$$L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let us give a remark about the manifolds $\mathcal{Y}^{\pm}$. The Killing form of $\mathfrak{g}$ is $B(X,Y) = \mathrm{tr}\,(XY)$, so that $B(X,X) = 2(-\xi_0^2 + \xi_1^2 + \xi_2^2)$. Consider in $\mathfrak{g}$ two light cones (forward and backward) C^+ and C^- defined by the inequalities $-\xi_0^2 + \xi_1^2 + \xi_2^2 < 0$, $\pm\xi_0 > 0$. These domains are invariant with respect to $\mathrm{Ad}\, G$. In the complexification $G^{\mathbb{C}}$ let us take the two sets $\exp(iC^{\pm})$. It turns out that

$$\mathcal{Y}^{\pm} = \mathcal{X} \cdot \exp(iC^{\pm}).$$

Moreover, it turns out that the subsets $\Gamma^{\pm} = G \cdot \exp(iC^{\pm})$ of $G^{\mathbb{C}}$ are *semigroups* (Olshanski).

Now let us consider a complexification $G^{\mathbb{C}}$ of the group G. It consists of complex matrice of the third order preserving the form $[x,\, y]$ in $\mathbb{C}^3$. Let us take the following matrices in $G^{\mathbb{C}}$:

$$\gamma_t = \mathrm{e}^{itL_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & -i\sinh t \\ 0 & i\sinh t & \cosh t \end{pmatrix},$$

$$\delta_t = \mathrm{e}^{itL_2} = \begin{pmatrix} \cos t & 0 & i\sin t \\ 0 & 1 & 0 \\ i\sin t & 0 & \cos t \end{pmatrix}. \tag{1.13}$$

The curves (1.12) are obtained when we multiply the point $x^0 = (0,0,1) \in \mathcal{X}$ by these matrices, i. e. $y_t = x^0\gamma_t$, $\omega_t = x^0\delta_t$.

Therefore, any point x in $\mathcal{Y}^\pm$ is $x^0\gamma_t g$, where $t > 0$ for $\mathcal{Y}^+$ and $t < 0$ for $\mathcal{Y}^-$, and any point x in $\Omega^\pm$ is $x^0\delta_t g$, where $0 < t < \pi/2$ for Ω^+ and $-\pi/2 < t < 0$ for Ω^-. Here g ranges over G.

Let us return to the G_1-orbits of the pairs $(0,\, \infty)$ and $(\infty,\, 0)$ which were deleted from $D\times D'$ and $D'\times D$ respectively. Under the map (1.3) the pairs $(0,\, \infty)$ and $(\infty,\, 0)$ go respectively to the points $\omega_{\pi/2} = (i,0,0) = ix^1$ and $\omega_{-\pi/2} = (-i,0,0) = -ix^1$, where $x^1 = (1,0,0)$. Therefore, the map (1.3) carries these G_1-orbits to the G-orbits of the points ix^1 and $-ix^1$. Both points $\pm x^1$ belong to the hyperboloid $[x,\, x] = -1$. It consists of two the sheets $\mathcal{L}^\pm$, so that $x^1 \in \mathcal{L}^+$ and $-x^1 \in \mathcal{L}^-$. Therefore the G-orbits are $i\mathcal{L}^\pm$. They lie on the boundary of the manifolds $\Omega^\pm$ respectively. Each of them can be identified with the Lobachevsky plane $\mathcal{L} = G_1/K_1 = G/K$.

All four complex manifolds (of real dimension 4) $\mathcal{Y}^\pm$, $\Omega^\pm$ are adjoint to the one-sheeted hyperboloid $\mathcal{X}$ (of real dimension 2). On their turn, each of the two manifolds Ω^+ and Ω^- are adjoint to one sheet (to $i\mathcal{L}^+$ and $i\mathcal{L}^-$) of the two-sheeted hyperboloid $[x,\, x] = -1$ (of real dimension 2). Relative to the Lobachevsky plane $i\mathcal{L}^\pm$ the manifold $\Omega^\pm$ is a *"complex crown"* (after Akhiezer–Gindikin).

Let us assign to any point x in the manifolds $\mathcal{Y}^\pm$, $\Omega^\pm$ its third coordinate $x_3 \in \mathbb{C}$.

Lemma 1.2 *Under the map $x \mapsto x_3$ the image of the manifold $\mathcal{Y}^\pm$ is the whole complex plane $\mathbb{C}$ with the cut $[-1,\, 1]$, the image of the manifold $\Omega^\pm$ is the whole complex plane with cuts $(-\infty,\, 1]$ and $[1,\, \infty)$.*

Proof. For a point $x \in \mathcal{X}^{\mathbb{C}}$ with coordinates z, w, see (1.3), we have

$$\frac{x_3+1}{x_3-1} = \frac{1+iz}{1-iz}\cdot\frac{1-iw}{1+iw}. \tag{1.14}$$

The function $z \mapsto \zeta = (1+iz)/(1-iz)$ maps the disk D onto the right half-plane $\mathrm{Re}\,\zeta > 0$, and its exterior D' onto the left half-plane $\mathrm{Re}\,\zeta < 0$. Therefore, if $(z,w) \in D\times D$ or $(z,w) \in D'\times D'$, then both fractions in the right-hand side of (1.14) range over either the left or the right half-plane. Their product ranges over the whole plane $\mathbb{C}$ with cut $(-\infty,\,0]$. If in addition $z \neq w$, then this product is not equal to 1. Hence x_3 ranges over $\mathbb{C}$ without $[-1,\,1]$.

If $(z,w) \in D \times D'$ or $(z,w) \in D' \times D$, then both fractions in the right-hand side of (1.14) range over different half-planes. Therefore, their product ranges over the whole of $\mathbb{C}$ with cut $[0,\,\infty)$, hence x_3 ranges over $\mathbb{C}$ without $(-\infty,\,-1]$ and $[1,\,\infty)$. Since we consider $\Omega^{\pm}$, but not $D \times D'$ and $D' \times D$, we have to exclude in (1.14) pairs (z,w) for which $w = 1/\overline{z}$. But it does not make the image smaller. Indead, if the first fraction in the right-hand side of (1.14) has value $r\mathrm{e}^{i\alpha}$, $-\pi/2 < \alpha < \pi/2$, then the second fraction has value $-r^{-1}\mathrm{e}^{i\alpha}$, so that their product is equal to $-\mathrm{e}^{2i\alpha}$. The intersection of the sets of these points over all α is empty. □

Let $M(y)$ be a holomorphic function on the manifold $\mathcal{Y}^{\pm}$ and let $N(x)$ be its limit values at the hyperboloid $\mathcal{X}$:

$$M(x) = \lim_{y \to x} M(y),\ \ y \in \mathcal{Y}^{\pm},\ x \in \mathcal{X}.$$

We shall assume that y tends to x "along the radius", i. e. if $y \in \mathcal{Y}^{\pm}$ and $x \in \mathcal{X}$ have horospherical coordinates (z,w) and (u,v) respectively, then

$$z = \mathrm{e}^{-t}u,\ \ w = \mathrm{e}^{-t}v \tag{1.15}$$

and $t \to \pm 0$. These equalities (1.15) give (for γ_t, see (1.13)):

$$y = x\gamma_t. \tag{1.16}$$

Lemma 1.3 *Let $M(y)$ depend only on y_3: $M(y) = N(y_3)$. By Lemma 1.2 the function $N(\lambda)$ is analytic on the plane $\mathbb{C}$ with cut $[-1,\,1]$. Then one has*

$$M(x) = N(x_3 \mp i0x_2).$$

Proof. Let y and x be connected by (1.15). Then by (1.16) we have

$$y_3 = -i \sinh t \cdot x_2 + \cosh t \cdot x_3.$$

Therefore, $y_3 = -it \cdot x_2 + x_3 + o(t)$, when $t \to 0$. Hence the lemma. □

Now let $M(\omega)$ be a holomorphic function on the manifold $\Omega^{\pm}$ and let $M(x)$ be its limit values on the hyperboloid $\mathcal{X}$:

$$M(x) = \lim_{y \to x} M(\omega).$$

Here we assume similarly that ω tends to x "along the radius", i. e. if $\omega \in \Omega^{\pm}$ and $x \in \mathcal{X}$ have horospherical coordinates (z,w) and (u,v) respectively, then

$$z = \mathrm{e}^{-t}u,\ w = \mathrm{e}^{t}v \tag{1.17}$$

and $t \to \pm 0$.

Lemma 1.4 *Let $M(\omega)$ depend only on ω_3: $M(\omega) = N(\omega_3)$. By Lemma 1.2 the function $N(\lambda)$ is analytic on the plane $\mathbb{C}$ with cuts $(-\infty, -1]$ and $[1, \infty)$. Then*

$$M(x) = N(x_3 \pm i0 \cdot x_1 x_3) \tag{1.18}$$

Proof. By (1.17) and (1.18) we have

$$\omega_3 = \frac{1 + uv}{i(\mathrm{e}^{-t}u - \mathrm{e}^{t}v)}.$$

Let us substitute here the expressions of u, v in terms of x, see (1.2). Taking into account the equality $x_1^2 + 1 = (x_3 + ix_2)(x_3 - ix_2)$, we obtain

$$\omega_3 = \frac{x_3}{\cosh t - i \sinh t \cdot x_1}. \tag{1.19}$$

When $t \to 0$, it behaves as $x_3(1 + itx_1)$ up to terms of order t^2. Hence the lemma. □

It is convenient to represent it using a cone in $\mathbb{C}^4$. Let us equip $\mathbb{C}^4$ with the bilinear form

$$[[x,\, y]] = -x_0 y_0 - x_1 y_1 + x_2 y_2 + x_3 y_3$$

(we add to vectors x in $\mathbb{C}^3$ the coordinate x_0). Let $\mathcal{C}$ be the cone in $\mathbb{C}^4$ defined by $[[x,\, x]] = 0$, $x \neq 0$. Then the complex hyperboloid $\mathcal{X}^{\mathbb{C}}$ is the section of the cone $\mathcal{C}$ by the hyperplane $x_0 = 1$. Looking at (1.3), consider the set $\mathcal{Z}$ of points

$$\zeta = \frac{1}{2}\,\left(i(z - w),\, z + w,\, i(1 - zw),\, 1 + zw\right), \tag{1.20}$$

where $z, w \in \mathbb{C}$. It is the section of the cone $\mathcal{C}$ by the hyperplane $-ix_2 + x_3 = 1$, i. e. the hyperplane $[x,\, \xi^0] = 1$, where $\xi^0 = (0, 0, , -i, 1)$. The map $\zeta \mapsto x = \zeta/\zeta_0$ maps $\mathcal{Z} \backslash \{z = w\}$ in $\mathcal{X}^{\mathbb{C}}$, it gives just the horospherical coordinates.

The manifolds (1.11) without the points corresponding to ∞, lie in $\mathcal{Z}$: in order to obtain $D \times D$ or $D' \times D'$, one has to the inequality $[\zeta,\, \overline{\zeta}] > 0$ to add the inequality $\mathrm{Im}\,(\zeta_3/\zeta_2) < 0$ or $\mathrm{Im}\,(\zeta_3/\zeta_2) > 0$, respectively, and in order to obtain $D \times D'$ or $D' \times D$, one has to the inequality $[\zeta,\, \overline{\zeta}] < 0$ to add the condition that the imaginary part of the determinant

$$\begin{vmatrix} \zeta_0 & \zeta_1 \\ \overline{\zeta}_0 & \overline{\zeta}_1 \end{vmatrix}$$

is less or greater than zero.

[The definition just given of the manifold $D \times D$ is the definition of the Cartan domain of type IV $D(p)$ for $p = 2$. Indeed, $D(p)$ is defined as follows: we equip $\mathbb{C}^n$, $n = p + 2$, with the form $[x,\, x] = -x_1^2 - \ldots - x_p^2 + x_{n-1}^2 + x_n^2$. Let $\xi^0 = (0, \ldots, 0, -i, 1)$. Then $D(p)$ consists of points $\zeta \in \mathbb{C}^n$ such that

$$[\zeta,\, \zeta] = 0,\;\; [\zeta,\, \overline{\zeta}] > 0, \quad [\zeta,\, \xi^0] = 1, \quad \mathrm{Im}\,(\zeta_n/\zeta_{n-1}) < 0.$$

Any point $\zeta \in D(p)$ can be written as

$$\zeta = (\zeta_1, \ldots, \zeta_p, \frac{a-1}{2i}, \frac{a+1}{2}), \; a = \zeta_1^2 + \ldots + \zeta_p^2,$$

where $\zeta_1, \ldots, \zeta_p$ satisfy the inequalities:

$$\zeta_1\overline{\zeta}_1 + \ldots \zeta_p\overline{\zeta}_p < \frac{a\overline{a}+1}{2} < 1,$$

so that $D(p)$ can be identified with a bounded domain in $\mathbb{C}^p$.]

Let us assign to a point $x = (x_0, x_1, x_2, x_3)$ in $\mathbb{C}^4$ the matrix

$$x = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - ix_1 \end{pmatrix}. \tag{1.21}$$

Its determinant is equal to $-[[x,\ x]]$. In particular, for a point $\zeta \in \mathcal{Z}$, given by (1.20), one gets the matrix

$$\zeta = i \begin{pmatrix} z & 1 \\ -zw & -w \end{pmatrix} = i \begin{pmatrix} 1 \\ -w \end{pmatrix} \begin{pmatrix} z & 1 \end{pmatrix}.$$

These matrices ζ are characterized by $\det \zeta = 0$, $\zeta_{12} = i$.

The group $G_1 \times G_1$ acts on the space of matrices (1.21): to an element $(g_1, g_2) \in G_1 \times G_1$ corresponds the linear transformation:

$$x \mapsto g_1^{-1} x g_2. \tag{1.22}$$

It is given by a *real* matrix of order 4. We obtain a homomorphism of the group $G_1 \times G_1$ onto the group $\mathrm{SO}_0(2,2)$. The kernel is the group of order 2 consisting of the pairs $(E,\ E)$, $(-E,\ -E)$. The diagonal of $G_1 \times G_1$, i. e. the set of pairs $(g,\ g)$, $g \in G_1$, goes to the subgroup $\mathrm{SO}_0(1,2) = G$, it preserves $x_0 = (1/2)\,\mathrm{tr}\, x$.

Consider the following action of $G_1 \times G_1$ on $\mathcal{Z}$:

$$\zeta \mapsto \frac{g_1^{-1}\zeta g_2}{-i(g_1^{-1}\zeta g_2)_{12}}$$

(apply first the linear action (1.22) and return then to the section $\mathcal{Z}$ along a line passing through the origin). This is the fractional linear action:

$$z \mapsto z \cdot g_2, \quad w \mapsto w \cdot g_1.$$

The homomorphism $G_1 \times G_1 \to \mathrm{SO}_0(2,2)$ can be extended to the complexifications (we obtain a homomorphism $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C})$). Let us take in the complexification $G_1^{\mathbb{C}} \times G_1^{\mathbb{C}}$ the pairs

$$\left(\mathrm{e}^{itL_0}, \mathrm{e}^{itL_0}\right) \text{ and } \left(\mathrm{e}^{-itL_0}, \mathrm{e}^{itL_0}\right).$$

Under the above homomorphism these pairs go to the matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & -i\sinh t \\ 0 & 0 & i\sinh t & \cosh t \end{pmatrix} \text{ and } \begin{pmatrix} \cosh t & i\sinh t & 0 & 0 \\ i\sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first one is obtained by bordering of the matrix γ_t, see (1.13). The second one appears in the proof of Lemma 1.4. Indeed, let us multiply a vector (row) $(1, x_1, x_2, x_3)$ in $\mathcal{C}$, such that the vector $x = (x_1, x_2, x_3)$ belongs to $\mathcal{X}$, by this matrix and then divide by the coordinate with index zero, we just get a vector $\omega \in \Omega^{\pm}$ whose horospherical coordinates are connected with the horospherical coordinates of x by (1.17), namely,

$$\omega = \frac{1}{\cosh t - ix_1 \cdot i\sinh t} \left(x_1 \cdot \cosh t + i\sinh t,\ x_2,\ x_3 \right). \tag{1.23}$$

It includes (1.19). The curve (1.23) with $x = x^0$, i. e. the curve $(i\tanh t, 0, 1/\cosh t)$, is in fact the curve ω_t, see (1.12), with another parameter.

§ 2. On the analytic continuation of spherical functions

First we consider spherical functions $\Psi_{\sigma,\varepsilon}(x)$, $\sigma = -1/2 + i\rho$, $\varepsilon = 0, 1$, on the hyperboloid $\mathcal{X}$ of the *continuous* series. We study the analytic continuation of these functions to the complex manifolds $\Omega^{\pm}$, defined in § 1.

In fact, we may consider the spherical functions $\Psi_{\sigma,\varepsilon}$ on the hyperboloid $\mathcal{X}$ with generic σ: $\sigma \in \mathbb{C}$, $\sigma \notin \mathbb{Z}$, not only with $\sigma = -1/2 + i\rho$.

These spherical functions $\Psi_{\sigma,\varepsilon}(x)$, $\sigma \in \mathbb{C}$, $\varepsilon = 0, 1$, were computed in [4], [5], they are linear combinations of Legendre functions of the first kind of $\pm x_3$ (x_3 being the third coordinate of x):

$$\Psi_{\sigma,\varepsilon}(x) = -\frac{2\pi}{\sin \sigma\pi} \left[P_\sigma(-x_3) + (-1)^\varepsilon P_\sigma(x_3) \right]. \tag{2.1}$$

The Legendre function $P_\sigma(z)$ is analytic in the complex plane $\mathbb{C}$ with cut $(-\infty, -1]$. At this cut, we define it as half the sum of the limit values form above and below:

$$P_\sigma(c) = \frac{1}{2} \left[P_\sigma(c + i0) + P_\sigma(c - i0) \right], \ c < -1.$$

Therefore, the combination $P_\sigma(-z) + (-1)^\varepsilon P_\sigma(z)$ is analytic in $\mathbb{C}$ with cuts $(-\infty, -1]$ and $[1, \infty)$. Precisely these cuts appear in Lemmas 1.2 and 1.4 in treating the complex manifolds $\Omega^{\pm}$ adjoint to $\mathcal{X}$.

So, if we want to continue the functions $\Psi_{\sigma,\varepsilon}(x)$ analytically, then naturally $\Omega^\pm$ should be taken for that purpose. Hence let us consider functions on $\Omega^\pm$ defined by the same formula (2.1) with x replaced with ω:

$$\Psi_{\sigma,\varepsilon}(\omega) = -\frac{2\pi}{\sin\sigma\pi}\left[P_\sigma(-\omega_3) + (-1)^\varepsilon P_\sigma(\omega_3)\right]. \tag{2.2}$$

These functions are analytic on $\Omega^\pm$. Let us determine their limit values at $\mathcal{X}$.

First consider the function $P_\sigma(\omega_3)$ on $\Omega^\pm$. By Lemma 1.4 its limit values at $\mathcal{X}$ are:

$$\lim P_\sigma(\omega_3) = \begin{cases} P_\sigma(x_3), & x_3 > -1, \\ P_\sigma(x_3 \mp i0x_1), & x_3 < -1, \end{cases} \tag{2.3}$$

when $\omega \to x$ and $\omega \in \Omega^\pm$. It follows from [1] 3.3 (10) that the function $P_\sigma(z)$ has the following limit values at the cut $(-\infty, -1]$:

$$P_\sigma(c \pm i0) = \mathrm{e}^{\pm i\sigma\pi} P_\sigma(-c) - \frac{2}{\pi}\sin\sigma\pi \cdot Q_\sigma(-c),$$

where $c < -1$ and Q_σ is the Legendre function of the second kind. Therefore, by (2.3) we have for $\omega \to x$, $\omega \in \Omega^\pm$ and $x_3 < -1$:

$$\lim P_\sigma(\omega_3) = \begin{cases} \mathrm{e}^{\mp i\sigma\pi} P_\sigma(-x_3) - \dfrac{2}{\pi}\sin\sigma\pi Q_\sigma(-x_3), & x_1 > 0, \\ \mathrm{e}^{\pm i\sigma\pi} P_\sigma(-x_3) - \dfrac{2}{\pi}\sin\sigma\pi Q_\sigma(-x_3), & x_1 < 0. \end{cases}$$

It can be written as

$$\lim P_\sigma(\omega_3) = P_\sigma(c) \mp \operatorname{sgn} x_1 \cdot i\sin\sigma\pi \cdot P_\sigma(-x_3).$$

We see that the limit values of $P_\sigma(\omega_3)$ as $\omega \to x$ coincide with $P_\sigma(x_3)$ for $x_3 > -1$ only. In order to obtain $P_\sigma(x_3)$ one has to take half the sum of the limit values from *both* manifolds Ω^+ and Ω^-.

Similarly this goes for $P_\sigma(-y_3)$.

Thus, the limit values at $\mathcal{X}$ of the function $\Psi_{\sigma,\varepsilon}(\omega)$ on $\Omega^\pm$ defined by (2.2) coincide with the spherical function $\Psi_{\sigma,\varepsilon}(x)$ for $-1 < x_3 < 1$ only. The spherical function $\Psi_{\sigma,\varepsilon}(x)$ *is even* in x_1, but the limit function $\lim \Psi_{\sigma,\varepsilon}(\omega)$ *does not*. In order to obtain the spherical function $\Psi_\sigma(x)$ from the function $\Psi_{\sigma,\varepsilon}(\omega)$, one has to use *both* manifolds $\Omega^\pm$ and to take half the sum of the limit values from Ω^+ and Ω^-:

$$\Psi_{\sigma,\varepsilon}(x) = \frac{1}{2}\sum_{\pm} \lim\, \Psi_{\sigma,\varepsilon}(\omega),$$

where the limit is taken when $\omega \to x$, $\omega \in \Omega^\pm$, $x \in \mathcal{X}$.

Now we consider the spherical functions $\Psi_{n,\pm}$ of the *holomorphic and anti-holomorphic discrete* series on $\mathcal{X}$. Here $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The signs "$+$" and "$-$" correspond to the holomorphic and the anti-holomorphic series respectively. These functions are expressed [4] in terms of Legendre functions of the second kind:

$$\Psi_{n,\pm}(x) = 4Q_n(x_3 \mp i0 \cdot x_2).$$

Comparing it with Lemma 1.3, we see that the spherical function $\Psi_{n,\pm}$ is continued analytically on the manifold $\mathcal{Y}^{\pm}$ as the function

$$\Psi_n(y) = 4Q_n(y_3).$$

References

1. A. Erdélyi, W. Magnus, F. Oberthettinger and F. G. Tricomi. Higher Transcendental Functions, McGraw-Hill, New York, 1953.

2. V. F. Molchanov. Quantization on the imaginary Lobachevsky plane, Funct. Anal. Appl., 1980, vol. 14, issue 2, 73–74.

3. V. F. Molchanov. Holomorphic discrete series for hyperboloids of Hermitian type. J. Funct. Anal., 1997, vol. 147, No. 1, 26–50

4. V. F. Molchanov. Canonical representations on a hyperboloid of one sheet. Preprint Math. Inst. University of Leiden, MI 2004-02, 56 p.

5. V. F. Molchanov. Canonical and boundary representations on a hyperboloid of one sheet. Acta Appl. Math., 2004, vol. 81, Nos. 1–3, 191–204.