# A STUDY OF PARTICULAR METHODS FOR THE APPROXIMATE CONSTRUCTION OF SOME REGULAR POLYGONS BY USING MATHEMATICA 3.0

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In this paper we study particular methods for the approximate construction of the regular pentagon, heptagon, enneagon, decagon and undecagon by using Mathematica 3.0. In the same way, we make a study of the errors of those methods whose construction is not exact, and we compare them with those obtained in the general methods of Archimedes and Bardin. We also include the coded algorithm in Mathematica 3.0 for the animation of each method.

## 1 Preliminaries

Let us $LE_n$ denote the exact longitude on the side of the regular polygon of $n$ sides inscribed in a circumference of radius 1, this is

$$LE_n = 2\sin\left(\frac{\pi}{n}\right) \quad \forall n \geq 3,$$

and $L_n$ the longitude on the side of the regular polygon of $n$ sides inscribed in the same circumference constructed by the corresponding particular algorithm.

In the other hand, let $d(P_1, P_2)$ be the euclidean distance between the points $P_1, P_2 \in \mathbf{R}^2$, and $C[P,r] = C[(a,b),r]$ the circumference centered on $P = (a,b)$ and radius $r$. In particular $C[(0,0),1]$ is the goniometric circumference and $C[P, d(M,N)]$ the circumference centered on $P$ and radius equal to the distance between $M$ and $N$.

## 2 Construction of the regular pentagon

### 2.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters $AB$ and $CD$ (as main diameters we understand the two perpendicular diameters that coincide with the coordinated axes).
2) Determinate the point $P_1$ of negative ordinate, which is result of the intersection of $C[(0,0),1]$ and $C[(1,0),1]$.
3) Determinate the point $P_2$ of negative abscissa, which is result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.
4) Determinate the point $P_3$ of negative abscissa, which is result of the intersection of $C[P_1, d(P_1,P_2)]$ and the diameter $CD$.
5) The distance $L_5 = d(B, P_3)$ is the side of the regular pentagon that we want to draw.

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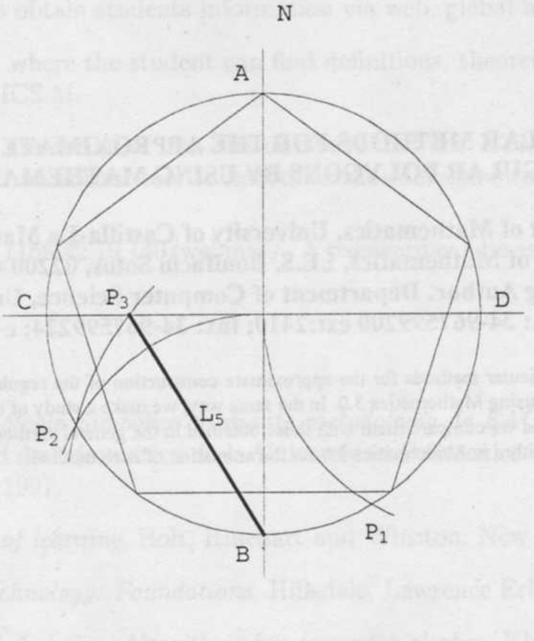


Figure 1: Construction of the pentagon

## 2.2 Analysis of the error

Let us prove, by using *Mathematica 3.0*, that above algorithm is exact, this is, that $L_5 = LE_5$.

The points $P_1$ and $P_2$ can be founded by means of

$$\mathrm{Solve}[\{x^2+y^2==1,(x-1)^2+y^2==1\},\{x,y\}]$$
$$\mathrm{Solve}[\{x^2+(y+1)^2==1,x^2+y^2==1\},\{x,y\}]$$

resulting

$$P_1=\left(\frac{1}{2},\frac{-\sqrt{3}}{2}\right) \quad \text{and} \quad P_2=\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right).$$

To calculate $P_3$ we previously need compute the distance between $P_1$ and $P_2$, that it can be obtain by using

$$\mathrm{distance}[\{a\_,b\_\},\{c\_,d\_\}] := \sqrt{(c-a)^2+(d-b)^2}$$
$$\mathrm{dis}(P1,P2) := \mathrm{distance}[\{1/2,-\sqrt{3}/2\},\{-\sqrt{3}/2,-1/2\}]$$
$$\mathrm{Solve}[\{(x-1/2)^2+(y+\sqrt{3}/2)^2==\mathrm{dis}(P1,P2)^2, y==0\},\{x,y\}]$$

outputting

$$P_3=\left(\frac{1-\sqrt{5}}{2},0\right).$$

Finally, the side $L_5$ can be calculated in the way

$$\mathrm{Lado5} := \mathrm{distance}[\{0,-1\},\{(1-\sqrt{5})/2,0\}]$$

that is equal to $LE_5$ as can be proved by means of the following algorithm

$$\mathrm{ExactSide}[n\_] := 2 * \mathrm{Sin}\left[\frac{\pi}{n}\right]$$
$$\mathrm{Abs}[\mathrm{ExactSide}[5]-\mathrm{Lado5}]$$

# 3 Construction of the regular heptagon

## 3.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters $AB$ and $CD$.
2) Determinate the point $P_1$ and $P_2$, result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.
3) Determinate the point $P_3$, result of the intersection of the straight line joining $P_1$ with $P_2$ and the diameter $AB$.
4) The distance $L_7 = d(P_1, P_3)$ is the side of the regular heptagon that we want to draw.

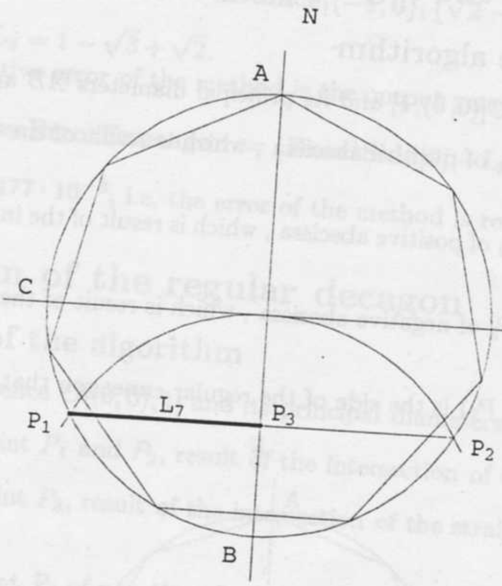


Figure 2: Construction of the heptagon

## 3.2 Analysis of the error

Let us start determinating the points $P_1$ and $P_2$

$$\mathbf{Solve}[\{\mathbf{x}^2 + \mathbf{y}^2 == 1, \mathbf{x}^2 + (\mathbf{y}+1)^2 == 1\}, \{\mathbf{x}, \mathbf{y}\}]$$

obtaining

$$P_1 = \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right) \quad \text{y} \quad P_2 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right),$$

and therefore

$$P_3 = \left(0, \frac{-1}{2}\right).$$

Hence, we can calculate $L_7$ by means of

$$\mathbf{Lado7} := \mathbf{distance}[\{0, -1/2\}, \{(-\sqrt{3}/2, -1/2\}]$$

resulting the value $L_7 = \sqrt{3}/2$.

We will obtain an estimate of the precision of the method starting from the *relative error* $(LE_7 - L_7)/LE_7$. The reason why we don't consider the error taking the absolute value of the difference $(LE_7 - L_7)$, is that we want to know if the value $L_7$ obtained by means of the algorithm approaches for excess or for defect to the exact longitude $LE_7$. In particular, if the relative error is positive then the approach is for defect, else it is for excess.

The code

$$\mathbf{RelativeErrorHeptagon := (ExactSide[7] - Lado7)/ExactSide[7]}$$

produces the value $2.00754 \cdot 10^{-3}$, i.e., the error of the method is roughly 0.2%.

# 4 Construction of the regular enneagon

## 4.1 Description of the algorithm

1) Trace the circumference $C[(0,0),1]$ and its principal diameters $AB$ and $CD$.
2) Determinate the point $P_1$ of positive abscissa , which is result of the intersection of $C[(0,0),1]$ and $C[(0,-1),1]$.
3) Determinate the point $P_2$ of positive abscissa , which is result of the intersection of $C[(0,1),d(A,P_1)]$ and the diameter $CD$.
4) Determinate the point $P_3$ of negative abscissa , which is result of the intersection of $C[P_2,d(P_2,B)]$ and the diameter $CD$.
5) The distance $L_9 = d(C,P_3)$ is the side of the regular enneagon that we want to draw.

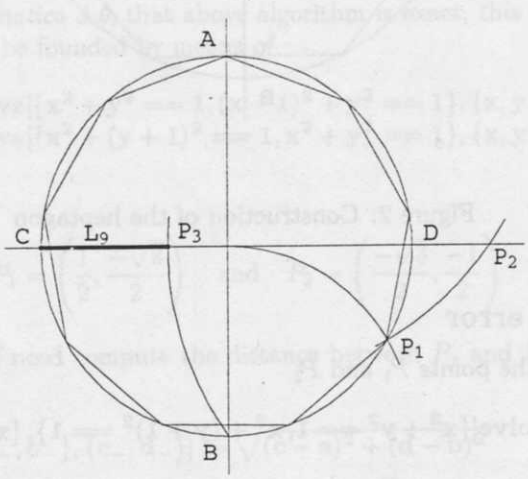


Figure 3: Construction of the enneagon

## 4.2 Analysis of the error

Let us calculate $P_1$ by means of the code

$$\mathbf{Solve[\{x^2 + y^2 == 1, x^2 + (y+1)^2 == 1\}, \{x, y\}]}$$

resulting

$$P_1 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right).$$

To determinate $P_2$ can be used the following algorithm

$$\mathbf{dis(A,P1) := distancia[\{0,1\}, \{\sqrt{3}/2, -1/2\}]}$$
$$\mathbf{Solve[\{x^2 + (y-1)^2 == dis(A,P1)^2, y == 0\}, \{x, y\}]}$$

which outputs $P_2 = \{\sqrt{2}, 0\}$.

On the other hand, we can find $P_3$ in the way

$$\mathbf{dis(P2, B) := distance[\{\sqrt{2}, 0\}, \{0, -1\}]}$$

$$\mathbf{Solve[\{(x - \sqrt{2})^2 + y^2 == dis(P2, B)^2, y == 0\}, \{x, y\}]}$$

obtaining $P_3 = \{\sqrt{2} - \sqrt{3}, 0\}$.

Finally, the code

$$\mathbf{Lado9 := distance[\{-1, 0\}, \{\sqrt{2} - \sqrt{3}, 0\}]}$$

allows us to calculate $L_9 = 1 - \sqrt{3} + \sqrt{2}$.

In this case the relative error of the method is the output generated by the input

$$\mathbf{RelativeErrorEnneagon := (ExactSide[9] - Lado9)/ExactSide[9]}$$

resulting the value $2.74477 \cdot 10^{-3}$, i.e, the error of the method is roughly 0.3%.

# 5 Construction of the regular decagon

## 5.1 Description of the algorithm

1) Trace the circumference $C[(0,0), 1]$ and its principal diameters $AB$ and $CD$.

2) Determinate the point $P_1$ and $P_2$, result of the intersection of $C[(0,0), 1]$ and $C[(0,-1), 1]$.

3) Determinate the point $P_3$, result of the intersection of the straight line joining $P_1$ with $P_2$ and the diameter $AB$.

4) Determinate the point $P_4$ of negative abscissa, result of the intersection of the straight line joining $P_3$ with $C$ and $C[P_3, d(P_3, B)]$

5) The distance $L_{10} = d(P_4, C)$ is the side of the regular decagon that we want to draw.

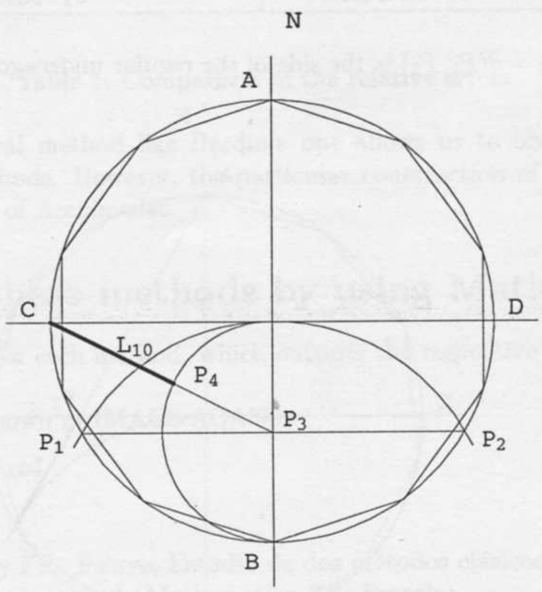


Figure 4: Construction of the decagon

### 5.2 Analysis of the error

We will check that the above algorithm allows us to construct the regular decagon in an exact way.

Observe that the points $P_1$, $P_2$ and $P_3$ coincide with its homologous ones in the case of the construction of the heptagon. To calculate $P_4$ we can use

$$\mathbf{Solve[\{x^2 + (y + 1/2)^2 == 1/4, y == -(1/2) * (x + 1)\}, \{x, y\}]}$$

obtaining

$$P_4 = \left(-\frac{\sqrt{5}}{5}, \frac{1}{10}(-5 + \sqrt{5})\right).$$

Therefore, $L_{10}$ can be determinated by means of

$$\mathbf{Lado10 := distance[\{-1, 0\}, \left\{-\frac{\sqrt{5}}{5}, \frac{1}{10}(-5 + \sqrt{5})\right\}]}$$

resulting the value $L_{10} = (-1 + \sqrt{5})/2$.

To prove that this algorithm is exact, it is enough to check that the following value vanishes

$$\mathbf{Abs[ExactSide[10] - Lado10]}$$

## 6 Construction of the regular undecagon

### 6.1 Description of the algorithm

1) Trace the circumference $C[(0,0), 1]$ and its principal diameters $AB$ and $CD$.
2) Determinate the point $P_1$ of positive abscissa , which is result of the intersection of $C[(0,0), 1]$ and $C[(0,-1), 1]$.
3) Determinate the point $P_2$, result of the intersection of the straight line joining $P_1$ with $A$ and the diameter $CD$.
4) The distance $L_{11} = d(P_1, P_2)$ is the side of the regular undecagon that we want to draw.

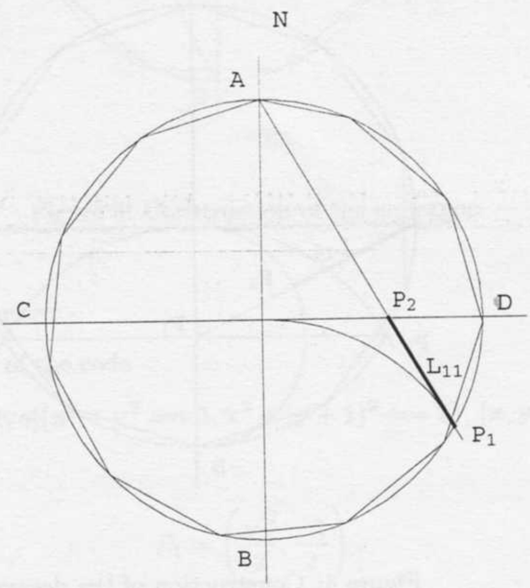


Figure 5: Construction of the undecagon

### 6.2 Analysis of the error

The point $P_1$ coincide with the point $P_2$ in the subsection 3.2, this is, $P_1 = (\sqrt{3}/2, -1/2)$.
Now, $P_2$ can be calculating by using

$$\mathbf{Solve}[\{\mathbf{y} == -\sqrt{3}\mathbf{x} + 1, \mathbf{y} == 0\}, \{\mathbf{x}, \mathbf{y}\}]$$

obtaining $P_2 = (\sqrt{3}/3, 0)$.
Finally we can obtain $L_{11}$ by means of the code

$$\mathbf{Lado11} := \mathbf{distance}[\{\sqrt{3}/3, 0\}, \{\sqrt{3}/2, -1/2\}]$$

which provides the value

$$L_{11} = \sqrt{\frac{1}{4} + \left(\frac{-\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\right)^2}.$$

Therefore, the relative error of the method

$$\mathbf{ErrorRelativoUndecagono} := (\mathbf{LadoExacto}[11] - \mathbf{Lado11})/\mathbf{LadoExacto}[11]$$

results $-2.46424 \cdot 10^{-2}$, i.e., in this occasion we approach for excess and the error is greater than the error of before cases.

## 7 Comparison with the general methods of Archimedes and of Bardin

Table 1 compares the relative errors of the particular algorithms for the construction of the pentagon, enneagon and undecagon with the relative errors that result of the layout of these polygons by using the general methods of Archimedes and of Bardin.

| $n$ | Rel. Error of the particular algorithm | Rel. error Archimedes | Rel. Error Bardin |
|---|---|---|---|
| 7 | $2.00754 \cdot 10^{-3}$ | $-1.62426 \cdot 10^{-3}$ | $2.86033 \cdot 10^{-4}$ |
| 9 | $2.74477 \cdot 10^{-3}$ | $-6.65798 \cdot 10^{-3}$ | $1.45450 \cdot 10^{-3}$ |
| 11 | $-2.46424 \cdot 10^{-2}$ | $-1.24955 \cdot 10^{-2}$ | $1.78029 \cdot 10^{-3}$ |

Table 1: Comparison of the relative errors

It is curious that a general method like Bardin's one allows us to obtain better results than the corresponding particular methods. However, the particular construction of the enneagon is better than that obtained by the method of Archimedes.

## 8 Animation of these methods by using Mathematica 3.0

We have made one program for each method, which outputs the respective guided construction (animation) of the regular polygon.
These programs will be shown at IMACS-ACA'99.

## References

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## TEACHING EFFICIENT MATHEMATICS

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The new generation of computer algebra systems, that have developed during the last 5 years, is characterized by thorough solving the problems and user friendliness. Solving mathematical problems that needed the numerous tedious calculations becomes now the fascinating work. The penetration of mathematics into various a spheres of knowledge swiftly intensifies. The XXI century would be the age of new chemical and biological technologies, moreover, it would be the age of new mathematical technologies. Contemporary systems of Computer Algebra are primary among such technologies. New mathematical technologies require an alteration of all systems of mathematical education.

The education of professional mathematicians must include a compulsory course titled "New mathematical technologies". This course should acquaint the fledgling professionals with the main technological tools and their usage, along with some test cases of their successful implementation.

The teaching of non-mathematicians is also to be changed radically. It should be based on intensive usage of the mathematical methods, especially with their new contemporary possibilities. The future specialist must have an opportunity to appreciate the potentialities of mathematics. So his mathematical education must train him for that. The high-effective computer systems have to be the necessary component of a such training. The mathematical education must consist of two parts. The first part is teaching to the language of mathematics. The second part is teaching to use the mathematical instrument, that is formulation of applied problem, solving, analysis, interpretation of solutions with the help of mathematical technologies.

The task of mathematicians is the creation of universal computer mathematical systems and special mathematical packets for each applied sphere. This is the way the system Mathematica acts. The mathematical packet for applied sphere is a laboratory for special experiments, where the results would be received with extreme rapidness and low expenses.