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Eigenvalues of the Berezin transform [1]

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We consider polynomial quantization on para-Hermitian symmetric spaces G/H with the pseudo-orthogonal group $G = \mathrm{SO}_0(p,q)$. We give explicit expressions of the Berezin transform in terms of Laplace operators. For that, we compute eigenvalues of the Berezin transform on irreducible finite dimensional subspaces of polynomials on G/H

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We consider polynomial quantization, a variant of quantization in the spirit of Berezin, on para-Hermitian symmetric spaces G/H with pseudo-orthogonal group $G = \mathrm{SO}_0(p,q)$. For all these spaces, the connected component H_e of the subgroup H containing the identity of G is the direct product $\mathrm{SO}_0(p-1,q-1) \times \mathrm{SO}_0(1,1)$, so that G/H is covered by G/H_e (with multiplicity 1, 2 or 4). The dimension of G/H is equal to $2n-4$, where $n = p+q$, the signature is $(n-2, n-2)$. We restrict ourselves to the spaces G/H that are G-orbits in the adjoint representation of G.

In polynomial quantization, covariant and contravariant symbols are polynomials on G/H. Explicit formulas for rank one para-Hermitian symmetric spaces were given in [1]. Our spaces G/H with group $G = \mathrm{SO}_0(p,q)$ have *rank two*.

One of the main formulas in polynomial quantization is the expression of the Berezin transform in terms of Laplace operators on G/H. In order to obtain this expression one needs to know eigenvalues of the Berezin transform on irreducible finite dimensional subspaces.

Thus this paper has two goals: for our spaces G/H with group $G = \mathrm{SO}_0(p,q)$, we first compute eigenvalues of the Berezin transform, and then we write explicit expressions of the Berezin transform in terms of Laplace operators.

Let us introduce in $\mathbb{R}^n$ the following bilinear form:

$$[x,y] = \sum_{i=1}^{n} \lambda_i x_i y_i,$$

where $\lambda_i = 1$ for $i \leqslant p$, $\lambda_i = -1$ for $i > p$, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are vectors in $\mathbb{R}^n$. The group $G = \mathrm{SO}_0(p,q)$ is the connected component of the identity

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in the group $\mathrm{SL}(n,\mathbb{R})$ preserving the form $[x,y]$. We consider that G acts on $\mathbb{R}^n$ from the right: $x\mapsto xg$, so that we write vectors $x\in\mathbb{R}^n$ in the row form. We consider the general case $p>1, q>1$.

Let us introduce on G/H *horospherical coordinates* ξ,η where ξ and η are vectors in $\mathbb{R}^{n-2}$ as follows.

Let $\mathcal{C}$ be the cone $[x,x]=0$, $x\neq 0$ in $\mathbb{R}^n$. The group G acts on it transitively. Let us take in the cone two points: $s^-=(1,0,\dots,0,-1)$, $s^+=(1,0,\dots,0,1)$, and consider the following sections of the cone:

$$\Gamma^-=\{x_1-x_n=2\},\quad \Gamma^+=\{x_1+x_n=2\}.$$

These sections intersect almost all generating lines of the cone. Therefore the linear action of the group G on the cone gives rise to corresponding linear-fractional actions on Γ^- and Γ^+. We introduce coordinates ξ on Γ^- and η on Γ^+ as follows: for points $u\in\Gamma^-$ and $v\in\Gamma^+$ we set:

$$\begin{aligned} u &= u(\xi)=(1+\langle\xi,\xi\rangle,\ 2\xi,\ -1+\langle\xi,\xi\rangle),\\ v &= v(\eta)=(1+\langle\eta,\eta\rangle,\ 2\eta,\ 1-\langle\eta,\eta\rangle),\end{aligned}$$

where $\langle\varphi,\psi\rangle$ denotes the following bilinear form in $\mathbb{R}^{n-2}$:

$$\langle\varphi,\psi\rangle=\sum_{i=2}^{n-1}\lambda_i\varphi_i\psi_i.$$

Notice that

$$[u,v]=-2N(\xi,\eta),$$

where

$$N(\xi,\eta)=1-2\langle\xi,\eta\rangle+\langle\xi,\xi\rangle\langle\eta,\eta\rangle.$$

These coordinates ξ, η on G/H have to satisfy the condition $N(\xi,\eta)\neq 0$.

We refer to [1] concerning covariant and contravariant symbols. The key moment is to construct explicitly an operator A, for which both covariant and contravariant symbols are written explicitly.

The quasiregular representation of G by translations on the space of polynomials on G/H is the multiplicity free direct sum of irreducible finite dimensional representations with highest weights $(k+l,k-l)$, where $k,l\in\mathbb{N}$, $k\geqslant l$, acting on subspaces $\mathcal{H}_{k,l}$. The weights are taken with respect to the secondary diagonal Abelian subalgebra of the Lie algebra of G. Here $\mathbb{N}=\{0,1,2,\dots\}$.

Further we use the notation:

$$a^{[r]}=a(a+1)\dots(a+r-1),\quad a^{(r)}=a(a-1)\dots(a-r+1).$$

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Theorem 2.6 *Let σ be the parameter labeling co- and contravariant symbols. The eigenvalues $b_{k,l}(\sigma)$ of the Berezin transform $\mathcal{B}_\sigma$ on $\mathcal{H}_{k,l}$ are given by*

$$b_{k,l}(\sigma)=\frac{(\sigma+n-2)^{[k]}}{\sigma^{(k)}}\cdot\frac{(\sigma+m+2)^{[l]}}{(\sigma+m)^{(l)}}\,.$$

In order to find these eigenvalues, it is sufficiently to trace some polynomial in $\mathcal{H}_{k,l}$. As such polynomial, we take the lowest vector

$$F = \frac{\langle \eta, \eta \rangle^l \{(\xi_2 - \xi_{n-1})\langle \eta, \eta \rangle - (\eta_2 - \eta_{n-1})\}^{k-l}}{N(\xi, \eta)^k}\,.$$

It is the contravariant symbol of the operator

$$A = \frac{1}{2^{k+l}\sigma^{(k)}(\sigma + m)^{(l)}} \left(\frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_{n-1}} \right)^{k-l} \circ \Delta^l\,,$$

where

$$\Delta = \sum_{k=2}^{n-1} \lambda_k \frac{\partial^2}{\partial \xi_k^2}\,.$$

In turn, this operator A has the covariant symbol just the polynomial

$$F_1 = b_{k,l}(\sigma) \cdot F.$$

Now we can write an expression of the Berezin transform $\mathcal{B}_\sigma$ in terms of Laplace operators, see also [2]. For our space G/H, we have two Laplace operators Δ_2 and Δ_4, they are differential operators of the second and of the fourth order, respectively.

Theorem 2.7 *Denote $m = (n-4)/2$. We have*

$$\begin{aligned} \mathcal{B}_\sigma \;&=\; \frac{\Gamma(\sigma + n - 2 + k)\Gamma(\sigma + 1 - k)}{\Gamma(\sigma + n - 2)\Gamma(\sigma + 1)} \times \\ &\times\; \frac{\Gamma(\sigma + m + 2 + l)\Gamma(\sigma + m + 1 - l)}{\Gamma(\sigma + m + 2)\Gamma(\sigma + m + 1)}\,, \end{aligned}$$

where k, l some variables. In fact, the right hand side depends on λ_2 and λ_4 only, where $\lambda_2 = 2(a_1 + a_2)$, $\lambda_2 = 16(a_1 a_2 - m a_1 + m^2 a_2)$ and $a_1 = k(k+n-3)$, $a_2 = l(l+1)$. Instead of λ_2 and λ_4 one has to substitute Δ_2 and Δ_4, respectively.

References

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