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Invariant subspaces of smooth functions on the upper light cone

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A description of closed subspaces of C^∞ functions on the upper light cone in $\mathbb{R}^3$ invariant with respect to dilatations and the group $\mathrm{SO}_0(1,2)$ is given

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Let G be a Lie group, X a smooth manifold with a transitive action of G. We assume that the action is smooth, and we write it as the *right* action. Let $\mathcal{E}(X) = C^\infty(X)$ be the space of all infinitely differentiable complex-valued functions on X equipped with the standard topology. It is a complete locally convex space. A closed linear subspace $\mathcal{H} \subset \mathcal{E}(X)$ is called an invariant subspace, if $\mathcal{H}$ is invariant with respect to translations τ_g by means of $g \in G$:

$$(\tau_g f)(x) = f(xg).$$

A problem of spectral synthesis is: for given G and X, to describe all invariant subspaces $\mathcal{H} \subset \mathcal{E}(X)$. In general, this problem is extremely difficult. At present one has a solution only for some special cases of G and X, see, for example, [1]. In this paper we consider a new case when this problem of spectral synthesis can be solved.

Introduce in the space $\mathbb{R}^3$ the bilinear form

$$[x, y] := x_0y_0 - x_1y_1 - x_2y_2,$$

where $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$ are vectors in $\mathbb{R}^3$. Let X be the upper light cone defined by $[x, x] = 0$, $x_0 > 0$.

Denote by $\widetilde{G}$ the group $\mathrm{SO}_0(1, 2)$, the connected component of the identity in the group of linear transformations of $\mathbb{R}^3$ preserving the form $[x, y]$. It acts on $\mathbb{R}^3$ from the right: $x \mapsto xg$, $g \in \widetilde{G}$. On the cone X it acts transitively. Denote by Γ the group of dilatations $\gamma(t) = e^tE$, $t \in \mathbb{R}$, and E being the identity matrix. It acts by multiplications: $x \mapsto x\gamma(t) = e^tx$. The group Γ preserves the cone X. Finally, we put $G = \Gamma \times \widetilde{G}$. We are looking for invariant subspaces in $\mathcal{E}(X)$ with respect to this group G.

Let K be the subgroup of G consisting of matrices

$$k(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Let $\mathcal{E}^{(n)}$, $n \in \mathbb{Z}$, denote a closed linear subspace of $\mathcal{E}(X)$ consisting of functions f such that

$$f(xk(\theta)) = e^{in\theta} f(x), \quad \theta \in \mathbb{R}.$$

Let $x^0 = (1, 1, 0)$ be an "initial point" in X. The map

$$\alpha_n : f(x) \mapsto \widetilde{f}(t) := f(x^0\gamma(t)) \tag{1}$$

is an isomorphism of the topological vector space $\mathcal{E}^{(n)}$ on $\mathcal{E}(\mathbb{R})$.

Definition 5 A closed linear subspace $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ is called an *invariant cell* if there exists an invariant subspace $\mathcal{H} \subset \mathcal{E}(X)$ such that $\mathcal{H}^{(n)} = \mathcal{H} \cap \mathcal{E}^{(n)}$.

In this case we say that the cell $\mathcal{H}^{(n)}$ corresponds to the invariant subspace $\mathcal{H}$. Notice that such an invariant subspace $\mathcal{H}$ is not necessarily unique.

An invariant subspace $\mathcal{H}$ is recovered by the family of all invariant cells $\mathcal{H}^{(n)}$, $n \in \mathbb{Z}$, corresponding to this $\mathcal{H}$, indeed, we have

$$\mathcal{H} = \text{closure} \sum_{n \in \mathbb{Z}} \mathcal{H}^{(n)}, \quad \mathcal{H}^{(n)} = \mathcal{H} \cap \mathcal{E}^{(n)}.$$

Our plan of describing invariant subspaces $\mathcal{H}$ is as follows: first we describe the structure of invariant cells at all, and then determine conditions under which a family of cells contains all cells corresponding to a single invariant subspace.

Theorem 4 *A closed linear subspace $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ is an invariant cell if and only if it is invariant with respect to the group Γ.*

Let L be the Lie operator of the group Γ:

$$(Lf)(x) := \left. \frac{d}{ds} f(x\gamma(s)) \right|_{s=0}.$$

For $\lambda \in \mathbb{C}$, $r \in \mathbb{N}$, denote by $V_{\lambda,r}^{(n)}$ a subspace of $\mathcal{E}^{(n)}$ consisting of functions f such that

$$(L - \lambda)^r f = 0.$$

The map α_n sends $V_{\lambda,r}^{(n)}$ to a subspace of $\mathcal{E}(\mathbb{R})$ of dimension r spanned by the functions

$$e^{\lambda t}, te^{\lambda t}, \ldots, t^{r-1}e^{\lambda t}.$$

The subspaces $V_{\lambda,r}^{(n)}$ are the simplest invariant cells. The following theorem describes the structure of all cells of $\mathcal{E}^{(n)}$.

Theorem 5 *For any nontrivial invariant cell $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$, $\mathcal{H}^{(n)} \neq \mathcal{E}^{(n)}$, there exists a unique finite or countable set $\sigma \subset \mathbb{C}$ together with a multiplicity function $\lambda \mapsto r_\lambda$ on σ with values in $\mathbb{N}$ such that*

$$\mathcal{H}^{(n)} = \text{closure} \sum_{\lambda \in \sigma} V^{(n)}_{\lambda, r_\lambda} .$$

Let us call the collection σ the *spectrum* of the invariant cell $\mathcal{H}^{(n)}$.

Suppose we have a sequence $n \mapsto \mathcal{H}^{(n)}$, $n \in \mathbb{Z}$, of invariant cells (not necessarily nontrivial). Let $\sigma(n)$ be the spectrum of $\mathcal{H}^{(n)}$ with multiplicities $r^{(n)}_\lambda$. The following theorem gives necessary and sufficient conditions for this sequence to correspond to a single invariant subspace; as above it means that there exists an invariant subspace $\mathcal{H} \subset \mathcal{E}(X)$ such that $\mathcal{H}^{(n)} = \mathcal{H} \cap \mathcal{E}^{(n)}$, $n \in \mathbb{Z}$.

Theorem 6 *A sequence $n \mapsto \mathcal{H}^{(n)}$, $n \in \mathbb{Z}$, corresponds to a single invariant subspace $\mathcal{H} \subset \mathcal{E}(X)$ if and only if the next conditions are satisfied:*

1°. *If $\lambda \notin \mathbb{Z}$, then the multiplicity $r^{(n)}_\lambda$ does not depend on n.*

2°. *If $\lambda = k \in \mathbb{Z}$ and $k \geqslant 0$, then the multiplicities $r^{(n)}_\lambda$ are constant with respect to n for n in the following intervals:*

$$(-\infty, -k-1], \quad [-k, k], \quad [k+1, +\infty),$$

moreover, the multiplicity $r^{(k+1)}_k$ must be equal $r^{(k)}_k$ or $r^{(k)}_k - 1$, and the multiplicity $r^{(-k-1)}_k$ must be equal $r^{(-k)}_k$ or $r^{(-k)}_k - 1$.

3°. *If $\lambda = k \in \mathbb{Z}$ and $k < 0$, then the multiplicities $r^{(n)}_\lambda$ are constant with respect to n for n in the following intervals:*

$$(-\infty, k], \quad [k+1, -k-1], \quad [-k, +\infty),$$

moreover, the multiplicity $r^{(k+1)}_k$ must be equal $r^{(k)}_k$ or $r^{(k)}_k - 1$, and the multiplicity $r^{(-k-1)}_k$ must be equal $r^{(-k)}_k$ or $r^{(-k)}_k - 1$.

References

1. S. S. Platonov. On describing invariant subspaces of the space of smooth functions on a homogeneous manifold. Acta Appl. Math., 2004, 327–338.