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Indicator systems related to para-Hermitian symmetric spaces [1]

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We present systems of differential equations to describe spaces of finite dimensional representations of the group $\mathrm{SL}(n,\mathbb{R})$ acting on polynomials on the Heisenberg group of dimension $2n-3$

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It is known [2] that finite-dimensional representations of the complex group $\mathrm{SL}(n,\mathbb{C})$ belonging to the principal *nondegenerate* series can be realized in some spaces of polynomials on the subgroup Z of lower unipotent matrices. Zhelobenko, see [2], Ch X, introduced systems of differential equations for description of these spaces. He called them *indicator systems*. Spaces of solutions of indicator systems are just spaces where above-mentioned representations act.

We consider finite-dimensional representations of the real group $\mathrm{SL}(n,\mathbb{R})$ in the principal *degenerate* series corresponding to the partition $n = 1 + (n-2) + 1$ of the number n. They are realized in polynomials on the subgroup Z of lower unipotent *block* matrices. Notice that this group Z is the Heisenberg group of dimension $2n-3$. In our paper [1] we presented indicator systems for finite-dimensional representations related to polynomial quantization on the para-Hermitian symmetric space G/H, where $H = \mathrm{GL}(n-1,\mathbb{R})$. Now we extend this result to more general finite-dimensional representations corresponding to the partition $n = 1+(n-2)+1$.

Let us write matrices g in G in the block form corresponding to the partition $n = 1 + (n-2) + 1$. Denote by Z and B subgroups of G consisting of matrices

$$z = \begin{pmatrix} 1 & 0 & 0 \\ t & E & 0 \\ c & s & 1 \end{pmatrix}, \quad b = \begin{pmatrix} p & * & * \\ 0 & q & * \\ 0 & 0 & r \end{pmatrix}, \tag{1.1}$$

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respectively where E is the $(n-2)\times(n-2)$ identity matrix, s is a row-vector in $\mathbb{R}^{n-2}$, t is a column-vector in $\mathbb{R}^{n-2}$, c is a number in $\mathbb{R}$, p, r are numbers in $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$, q is a matrix in $\mathrm{GL}(n-2,\mathbb{R})$. The inverse for z is

$$z^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -t & E & 0 \\ \widehat{c} & -s & 1 \end{pmatrix},$$

where $\widehat{c} = st - c$. Let dz denote an invariant measure on Z:

$$dz = dc\, ds_2 \dots ds_{n-1}\, dt_2 \dots dt_{n-1}.$$

Almost every matrix $g \in G$ can be written as a product (the Gauss decomposition):

$$g = bz.$$

Let $S(Z)$ be the space of polynomials on Z. We denote $\mathbb{N} = \{0, 1, 2, \dots\}$. Let us take $l, m \in \mathbb{N}$. A representation $T_{l,m}$ of the group G acts on some subspace $V_{l,m}$ of the space $S(Z)$, see below, by the formula

$$\left(T_{l,m}(g)f\right)(z) = \frac{\widetilde{r}^{\,l}}{\widetilde{p}^{\,m}}\, f(\widetilde{z}),$$

where $\widetilde{z}$, $\widetilde{r}$, $\widetilde{p}$ are given by the Gauss decomposition of zg:

$$zg = \widetilde{b}\widetilde{z}.$$

The space $V_{l,m}$ contains the function 1 identically equal to the unit as a cyclic vector. The representation $T_{l,m}$ is irreducible, its lowest vector is 1, its highest vector is $c^l\,\widehat{c}^m$, the highest weight is $(l, 0, \dots, 0, -m)$, and the dimension is equal to

$$d_{l,m} = \frac{2m+n-1}{n-1}\binom{l+n-2}{l}\binom{m+n-2}{m}.$$

For $1 \leqslant i, j \leqslant n$, let E_{ij} denote a "matrix unit it is the matrix with 1 on the (i,j)th entry and zero on all other entries.

Matrices E_{i1} and E_{ni}, $i = 2, \dots, n-1$, are generators in the Lie algebra of the group Z. Let L_i and D_i be corresponding infinitesimal operators of left translations on the group Z. They are differential operators

$$L_i = \frac{\partial}{\partial t_i}, \quad D_i = t_i \frac{\partial}{\partial c} + \frac{\partial}{\partial s_i}, \quad i = 2, \dots, n-1.$$

Consider in the space $S(Z)$ a system of differential equations:

$$L_i^{m+1} f = 0, \quad D_i^{l+1} f = 0, \quad i = 2, \dots, n-1. \tag{1.2}$$

Let us call it, following Zhelobenko [2], the *indicator system*.

Theorem 1.1 *The space $V_{l,m}$ is precisely the space of all solutions of the indicator system* (2).

A crucial place in the proof of the theorem is an integral presentation of polynomials $f(z)$ in $V_{l,m}$:

$$f(z) = \int_Z K_{l,m}(z,\zeta) F(\zeta)\, d\zeta, \tag{1.3}$$

here F is a distribution on the group Z concentrated at the identity E of Z. For this point E we have $c = 0$, $s = 0$, $t = 0$. The kernel $K_{l,m}(z,\zeta)$, $z, \zeta \in Z$, has the following expression. Let z has parameters c, s, t, see (1), and ζ has parameters a, u, v, let J be the $(n-2) \times (n-2)$ diagonal matrix with the diagonal $\{-1, 1, \ldots, 1\}$, then

$$K_{l,m}(z,\zeta) = (1 - sJv + \widehat{a}\, c)^l \; (1 - uJt + \widehat{c}\, a)^m \, .$$

Thus, the kernel $K_{l,m}(z,\zeta)$ is a generating function for polynomials in $V_{l,m}$.

In particular, if in (3) the distribution F is the delta function $\delta(z)$ concentrated at the point E, then the polynomial $f(z)$ is the lowest vector 1.

In [1] we studied representations $T_{l,m}$ with $l = m$, these representations are used in the construction of polynomial quantization on G/H.

References

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