

ON SOME PROBLEMS OF APPROXIMATION ON NONCOMPACT SYMMETRIC SPACES

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1 Introduction

The modern technique of harmonic analysis on symmetric spaces makes possible to extend a lot of problems of classical harmonic analysis to symmetric spaces. In particular we can study the problems of function approximation on symmetric spaces.

Let X be a Riemannian symmetric space. Let $L_p(X)$, $1 \leq p < \infty$, be the set of all measurable functions f defined on X , for which the norm

$$\|f\|_p := \left(\int_X |f(x)|^p dx \right)^{1/p}$$

is finite, dx being the Riemannian measure on X . Let us denote by $L_\infty(X)$ the set of all continuous bounded functions on X , endowed with the norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

In classical case X is the circle S^1 or \mathbb{R} , the apparatus of approximation being the set of trigonometric polynomials or the set of entire functions of exponential type respectively (see [1], [2]). Let $\mathcal{P}_\nu = \mathcal{P}_\nu(X)$ be the set of all trigonometric polynomials of degree $\leq \nu$ for $X = S^1$ or the set of all entire functions of exponential type $\leq \nu$ for $X = \mathbb{R}$. The best approximation of a function $f(x) \in L_p(X)$ in L_p -metric is obtained by

$$E_\nu(f)_p := \inf_{\Phi \in \mathcal{P}_\nu} \|f - \Phi\|_p.$$

One of the main problems of approximation theory is to determine relations between the degree of decrease $E_\nu(f)_p$ as $\nu \rightarrow \infty$ and the intrinsic properties of f (its smoothness, its modulus of continuity, etc.) By means of direct Jackson-type theorems upper bounds of $E_\nu(f)$ by the continuity modulus of f have been found. Another result of the approximation theory is the description of certain function classes in terms of the best approximations and converse approximation theorems of which important elements are the Bernstein-type inequalities.

The problems of approximation on a compact symmetric space X of rank 1 by the spherical harmonic polynomials have been active studied in the recent years, especially the case when X is the n -dimensional sphere S^n . For compact symmetric space of rank 1 X the spherical harmonic polynomials are defined as linear combinations of eigenfunctions of the Laplace — Beltrami operator \mathcal{D} , in the case $X = S^n$ the spherical harmonic polynomials are the classical spherical polynomials. It is not my aim here to discuss these investigations, so I refer to the papers of Nikolskii and Lizorkin [3], Ragozin [4], Rustamov [5], Platonov [6,7,8], Luoqing [9], Kamzolov [10], Ivanov [11]. The reader can find there a bibliography of recent researches on this subject.

We consider the case when X is a Riemannian noncompact symmetric space of rank 1. For this case there are only a few results. A general approach to the approximation on an arbitrary Riemannian manifold X was developed by Lizorkin in [12], but the entire vectors of finite degree, which were used by him as apparatus of approximation, are not functions on X . Another possible apparatus of approximation on Riemannian manifold is the set of functions from $L_2(X)$ with bounded spectrum. The problems of the approximation theory on the n -dimensional Lobachevsky space by the functions with bounded spectrum was considered by Lizorkin and Petrova in [13], [14].

Further we shall consider certain problems of approximation theory in L_2 -metric on arbitrary Riemannian symmetric space of rank 1 by the functions with bounded spectrum (see [15]). The main subjects are: direct Jackson-type theorems, the description of Nikolskii — Besov spaces in terms of the best approximations, Bernstein-type inequalities. The special case of noncompact symmetric spaces of rank 1 is the n -dimensional Lobachevsky space and a lot of of these results are generalization of the results from [13], [14].

2 Functions with bounded spectrum and Bernstein-type inequalities

Any Riemannian symmetric space X can be realized as the quotient space G/K , where G is a real semisimple connected Lie group with finite center, K is a maximal compact subgroup of G . Let $G = NAK$ be the Iwasawa decomposition of G , M be the centralizer of A in K , $B = K/M$. By dx we denote the element of G -invariant measure on X , by db and dk the elements of K -invariant normalized measures on B and K respectively. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ and \mathfrak{n} be the Lie algebras of the Lie groups G, K, A and N respectively, \mathfrak{a}^* meaning the real dual space of \mathfrak{a} . Let Σ be the set of all restricted roots ($\Sigma \subset \mathfrak{a}^*$), Σ^+ be the set of all positive restricted roots,

$$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha.$$

By \langle, \rangle we denote the Killing form on \mathfrak{g} . This form is positive defined on \mathfrak{a} . For each $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we define

$$\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle.$$

\mathfrak{a}^* and \mathfrak{a} can be identified by the correspondence $\lambda \rightarrow H_\lambda$. Let

$$\mathfrak{a}_+^* = \{ \lambda \in \mathfrak{a}^* : \langle \alpha, \lambda \rangle > 0 \quad \forall \alpha \in \Sigma^+ \}$$

be the positive Weyl chamber.

For $g \in G$ by $A(g) \in \mathfrak{a}$ we denote the unique element such that

$$g = n \cdot \exp A(g) \cdot u,$$

where $u \in K, n \in N$. For $x = gK \in X = G/K$ and $b = kM \in B = K/M$ we define

$$A(x, b) = A(k^{-1}g).$$

By $C_c(X)$ we denote the set of all continuous complex-valued functions on X with compact support.

The Fourier transform of an arbitrary function $f \in C_c(X)$ is the function on $\mathfrak{a}^* \times B$ defined by the formula

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)A(x, b)} dx, \quad \lambda \in \mathfrak{a}^*, \quad b \in B$$

(see [16], [17]). Then the Plancherel formula

$$\int_X |f(x)|^2 dx = \int_{\mathfrak{a}_+^* \times B} |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \tag{1}$$

holds, where $c(\lambda)$ is the Harish-Chandra's c -function, $d\lambda$ is the Lebesgue measure on \mathfrak{a}^* properly normalized.

The Fourier transform $f(x) \rightarrow \tilde{f}(\lambda, b)$ uniquely extends from $C_c(X)$ to an isomorphism of the Hilbert space $L_2(X)$ to the Hilbert space $L_2(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2} d\lambda db)$ with the Plancherel formula still remaining in force.

A function $f \in L_2(X)$ is called function with bounded spectrum of type $\nu > 0$ if $\tilde{f}(\lambda, b) = 0$ for $|\lambda| > \nu$, where $|\lambda| = \langle \lambda, \lambda \rangle^{1/2}$. We denote by \mathcal{J}_ν the set of all this functions. It can be proved that $\mathcal{J}_\nu \subset C^\infty(X)$.

From here on let X be a noncompact Riemannian symmetric space of rank 1. In this case $\dim \mathfrak{a}^* = 1$. We choose the basis vector $H_0 \in \mathfrak{a}^*$ such that $H_0 \in \mathfrak{a}_+^*$ and $|H_0| = 1$. We will identify \mathfrak{a}^* with \mathbb{R} by

means of the correspondence $t \leftrightarrow tH_0, t \in \mathbb{R}$. By this identification the set a_+^* correspond to the set of positive real numbers. By $d(x, y)$ we denote the distance from x to y , where $x, y \in X$. Let

$$\sigma(x; t) = \{y \in X : d(y, x) = t\}$$

be the sphere in X of radius t and center x . By $d\mu_x(y)$ we understand the surface element of $\sigma(x; t)$ and let $|\sigma(t)|$ be volume of $\sigma(x; t)$.

For $f \in C_c(X)$ we introduce the function $S_t f$

$$(S_t f)(x) = \frac{1}{|\sigma(t)|} \int_{\sigma(x; t)} f(y) d\mu_x(y), \quad t > 0.$$

The operator S_t is called averaging operator. By continuity S_t can be extended to $L_2(X)$ and moreover

$$\|S_t f\|_2 \leq \|f\|_2 \quad \forall t > 0, \quad \forall f \in L_2(X),$$

where $\|\cdot\|_2$ is the norm in $L_2(X)$.

We define the finite differences $\Delta_t^k f, k = 0, 1, 2, \dots$, of a function $f \in L_2(X)$ with a step $t > 0$ by the rules

$$\begin{aligned} \Delta_t^0 f(x) &:= f(x), & \Delta_t^1 f(x) &:= \Delta_t f(x) := f(x) - S_t f(x), \\ \Delta_t^k f(x) &:= \Delta_t(\Delta_t^{k-1} f(x)). \end{aligned}$$

We can also write

$$\Delta_t^k f(x) = (I - S_t)^k f(x)$$

where I is the identity operator. Let \mathcal{D} be the Laplace — Beltrami operator on X .

Lemma 1 *Let $\Phi \in \mathcal{J}_\nu, \nu \geq 1, t, 0, k = 1, 2, \dots$. Then there exist positive constants c_1, c_2 ($c_1 = c_1(X, k), c_2 = c_2(X, k)$) such that*

$$\|\mathcal{D}^k \Phi\|_2 \leq c_2 \nu^{2k} \|\Phi\|_2; \tag{2}$$

$$\|\Delta_t^k \Phi\|_2 \leq c_3 (\nu t)^{2k} \|\Phi\|_2, \tag{3}$$

The proof see in [15]. The inequalities (2), (3) and their analogs are called the Bernstein type inequalities. They play an important role for the converse approximayion theorems.

3 Direct Jackson–type theorems

For $f \in L_2(X)$ let

$$\omega_k(f, \delta)_2 := \sup_{0 < t \leq \delta} \|\Delta_t^k f\|_2, \quad \delta > 0.$$

$\omega_k(f, \delta)_2$ is called the spherical continuity modulus of order k . By

$$E_\nu(f)_2 := \inf_{\Phi \in \mathcal{J}_\nu} \|f - \Phi\|_2.$$

we define the best approximation of a function $f \in L_2(X)$ by functions with bounded spectrum of type ν in L_2 .

The following theorems are called direct Jackson–type theorems.

Theorem 1 *There exists positive constant $c_4 = c_4(X, k), k = 1, 2, \dots$, such that for $f \in L_2(X), \nu \geq 1$*

$$E_\nu(f) \leq c_4 \omega_k(f, 1/\nu)_2, \tag{4}$$

Let $s \in \mathbb{N} = 1, 2, \dots, \mathcal{D}$ be the Laplace — Beltrami operator on X . In general $f \in L_2(X)$ the element $\mathcal{D}f$ is a distribution on X .

Theorem 2 *Let $f, \mathcal{D}f, \dots, \mathcal{D}^s f \in L_2(X)$. Then there exists positive constant $c_5 = c_5(X, k, s)$ such that for $\nu \geq 1$*

$$E_\nu(f) \leq c_5 \frac{\omega_k(\mathcal{D}^s f, 1/\nu)_2}{\nu^{2s}}.$$

The main elements of the proof of Theorem 1 are the following Lemmas 2 and 3. Using the Harish-Chandra's formula for spherical functions (see [17]) for $\lambda \in \mathfrak{a}^*$ we denote by $\varphi_\lambda(x)$ the spherical function

$$\varphi_\lambda(x) = \int_K e^{(i\lambda + \rho)A(kg)} dk, \quad x = gK \in X = G/K.$$

The spherical function $\varphi_\lambda(x)$ depends only on the distance between the points x and $o = eK \in X = G/K$, e being the unit element of G , and we can write

$$\varphi_\lambda(x) = \bar{\varphi}_\lambda(t), \quad t = d(x, o).$$

Lemma 2 For every $f \in L_2(X)$

$$\widetilde{S_t f}(\lambda, b) = \bar{\varphi}_\lambda(t) \cdot \tilde{f}(\lambda, b)$$

where $f \mapsto \tilde{f}$ is the Fourier transform on X .

Lemma 3 For all $\lambda \in \mathfrak{a}^*$, $t > 0$ the following inequalities hold:

- 1) $|\bar{\varphi}_\lambda(t)| \leq 1$;
- 2) $1 - \bar{\varphi}_\lambda(t) \leq t^2(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)$;
- 3) $1 - \bar{\varphi}_\lambda(t) \geq c$ if $\lambda t \geq 1$ where $c = c(X) > 0$ is a constant.

Proof of Theorem 1.

It follows from the Plancherel formula that

$$E_\nu^2(f)_2 = \int_\nu^\infty \int_B |\tilde{f}(\lambda, b)|^2 d\mu(\lambda) db, \tag{5}$$

where $d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda$. From Lemma 2 it follows that

$$1 - \bar{\varphi}_\lambda(1/\nu) \geq c$$

for $\lambda \geq \nu$. Then from (3), Lemma 3 and the Plancherel formula we obtain

$$\begin{aligned} E_\nu^2(f)_2 &\leq c^{-2k} \int_\nu^\infty \int_B (1 - \bar{\varphi}_\lambda(1/\nu))^{2k} |\tilde{f}(\lambda, b)| d\mu(\lambda) db \leq \\ &\leq c^{-2k} \int_0^\infty \int_B (1 - \bar{\varphi}_\lambda(1/\nu))^{2k} |\tilde{f}(\lambda, b)| d\mu(\lambda) db = \\ &= c^{-2k} \|(I - S_{1/\nu})^k f(x)\|_2^2 \leq c^{-2k} \omega_k^2(f, 1/\nu)_2, \end{aligned}$$

Hence

$$E_\nu(f) \leq c_4 \omega_k(f, 1/\nu)_2$$

with $c_4 = c^{-k}$.

The proof of Lemmas 2 - 3 and Theorem 2 see in [15].

4 Nikolskii — Besov spaces

Let $r > 0$, k and s be any nonnegative integers such that $2k > r - 2s > 0$. By definition a function $f(x) \in H_2^r = H_2^r(X)$ if $f, \mathcal{D}f, \dots, \mathcal{D}^s f \in L_2(X)$ and

$$\omega_k(\mathcal{D}^s f, \delta)_2 \leq A_f \delta^{r-2s} \quad \forall \delta > 0$$

for a positive constant A_f . For $f \in H_2^r$ by $h_2^r(f)$ we denote the seminorm

$$h_2^r(f) := \sup_{\delta > 0} \frac{\omega_k(\mathcal{D}^s f, \delta)_2}{\delta^{r-2s}}.$$

H_2^r is a Banach space with respect to the norm

$$\|f\|_{H_2^r} = \|f\|_2 + h_2^r(f). \tag{6}$$

The next theorem gives the description of the spaces H_2^r in terms of the best approximations.

Theorem 3 If $f \in H_2^r$ then for $\nu \geq 1$

$$E_\nu(f)_2 \leq c_6 \frac{h_2^r(f)}{\nu^r}.$$

Conversely, if $f \in L_2(X)$ and for $\nu \geq 1$ there exists a constant $A = A(f)$ such that

$$E_\nu(f)_2 \leq \frac{A}{\nu^r},$$

then $f \in H_2^r$ and

$$\|f\|_{H_2^r} \leq c_7(\|f\|_2 + A).$$

Here c_6, c_7 are constants, independent of f but which can depend on X, k, r, s .

Corollary 1 The space H_2^r does not depend on the $k, s \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ such that $2k > r - 2s > 0$.

These results can be transferred to Nikolskii — Besov spaces $B_{2q}^r = B_{2q}^r(X)$, $r > 0, 1 \leq q \leq \infty$. Let $k, s \in \mathbb{Z}_+$ be any such that $2k > r - 2s > 0$. By definition a function $f \in B_{2q}^r$ if $f, \mathcal{D}f, \dots, \mathcal{D}^s f \in L_2(X)$ and the seminorm $b_{2q}^r(f)$ is finite, where

$$b_{2q}^r(f) = \begin{cases} \left(\int_0^\infty \frac{\omega_k^q(\mathcal{D}^s f, \delta)_2}{\delta^{(r-s)q+1}} d\delta \right)^{1/q} & \text{with } 1 \leq q < \infty, \\ \sup_{\delta > 0} \frac{\omega_k(\mathcal{D}^s f, \delta)}{\delta^{r-s}} & \text{with } q = \infty. \end{cases}$$

B_{2q}^r is a Banach space with respect to the norm

$$\|f\|_{B_{2q}^r} = \|f\|_2 + b_{2q}^r(f). \tag{7}$$

Note that $B_{2\infty}^r = H_2^r$.

Theorem 4 Let $a > 1$ be arbitrary real number (for example we can take $a = 2$). A function $f \in L_2(X)$ belongs to the space B_{2q}^r if and only if $\tilde{b}_{2q}^r(f) < \infty$, where

$$\tilde{b}_{2q}^r(f) = \begin{cases} \sum_{n=0}^\infty a^{nrq} E_{a^n}^q(f)_2 & \text{with } q < \infty \\ \sup_{n=0,1,2,\dots} a^{nr} E_{a^n}(f)_2 & \text{with } q = \infty. \end{cases}$$

Moreover the norms (7) and

$$\|f\|_2 + \tilde{b}_{2q}^r(f)$$

are equivalent.

The proofs of Theorems 3 and 4 follow the scheme of corresponding theorems in [2] where the case $X = \mathbb{R}^n$ is considered. The main elements of proof are Theorem 2 and Bernstein type inequalities (2) — (3). See [15] for details.

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References

- [1] Ahiezer, N.I. (1965) *Lecture on Approximation Theory*. Nauka, Moscow (in Russian).
- [2] Nikolskii, S.M. (1977) *Approximation of Functions of many variables and Embedding Theorems*. Nauka, Moscow (in Russian).
- [3] Nikolskii, S.M., Lizorkin, P.I. (1989) Approximation on the sphere — A survey, *Banach Center Publications*, Vol. 22, pp. 281-292.

- [4] Ragozin, D.L. (1970) Polynomial approximation on compact manifolds and homogeneous spaces, *Trans. Amer. Math. Soc.*, **Vol. 150**, no. 1, pp. 41–54.
- [5] Rustamov, H.P. (1991) On approximation of functions on a sphere, *Dokl. AN USSA*, **Vol. 320**, no. 6, pp. 1319–1325 (in Russian).
- [6] Platonov, S.S. (1995) On approximation on compact symmetric spaces of rank 1, *Dokl. AN (Russia)*, **Vol. 342**, no. 4, pp. 455–457 (in Russian).
- [7] Platonov, S.S. (1997) Approximation on compact symmetric spaces of rank 1, *Matem. Sbornik* (to appear).
- [8] Platonov, S.S. (1966) On Nikolskii — Besov classes on compact symmetric spaces of rank 1, *Trudy Petrozavodskogo Univ. Ser. Matem.*, **Vol. 3**, pp. 153–172 (in Russian).
- [9] Luoqing, Li (1994) Riesz means on compact Riemannian symmetric spaces, *Math. Nachr.*, **Vol. 168**, pp. 227–242.
- [10] Kamzolov, A.I. (1974) On interpolational Riesz formula and Bernstein inequalities for functions on homogeneous spaces, *Mat. Zametki*, **Vol. 15**, no. 6, pp. 967–978 (in Russian).
- [11] Ivanov, V.A. (1983) On the Bernstein — Nikolskii and Favard inequalities on compact homogeneous spaces of rank 1, *Uspehi Mat. Nauk*, **Vol. 38**, no. 3, pp. 179–180 (in Russian).
- [12] Lizorkin, P.I. (1991) Approximation on Riemannian manifold, *Trudy MIRAN*, **Vol. 200**, pp. 222–235 (in Russian).
- [13] Lizorkin, P.I. (1992) Direct and converse approximation theorems for functions on the Lobachevskii space, *Trudy MIRAN*, **Vol. 194**, pp. 120–147 (in Russian).
- [14] Petrova, T.V. (1992) Approximation on the hyperboloid in L_2 -metric, *Trudy MIRAN*, **Vol. 194**, pp. 215–228 (in Russian).
- [15] Platonov, S.S. Approximations of functions in L_2 -metric on noncompact symmetric spaces, *St. Petersburg Math. J.* (to appear).
- [16] Helgason, S. (1970) A duality for symmetric spaces with applications to group representations, *Adv. in Math.*, **Vol. 5**.
- [17] Helgason, S. (1984) *Groups and Geometric Analysis*, Academic Press, New-York.