Representations on distributions on the Lobachevsky plane concentrated at the boundary

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In [4], the so-called canonical representations on the Lobachevsky plane were introduced – for the case when they are unitary. They are used for the construction of quantization etc. We consider them in a more general aspect and study their action on distributions concentrated at the boundary. It turns out that this action is diagonalizable. We give explicit expressions for distributions in irreducible constituents. The diagonalizability in question was discovered in [1] for para-Hermitian symmetric spaces.

1 Canonical representations on the Lobachevsky plane

Let us realize the Lobachevsky plane as the unit disk $D: z\bar{z} < 1$ on z-plane. The group $G = \mathrm{SU}(1,1)$ acts on D by fractional linear transformations

$$z \mapsto \tilde{z} = z \cdot g = \frac{az + \bar{b}}{bz + \bar{a}},\ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

(the group of motions of the Lobachevsky plane). Let Γ be the unit circle $z\bar{z} = 1$ and $\bar{D} = D \cup \Gamma$.

For $\lambda \in \mathbb{C}$, let R_λ be the representation of G acting on $\mathcal{D}(\bar{D})$ by the formula:

$$(R_\lambda(g)f)(z) = f(z \cdot g)|bz + \bar{a}|^{-2\lambda - 4}$$

It preserves the following sesqui-linear (Hermitian for $\lambda \in \mathbb{R}$) form

$$(f_1, f_2)_\lambda = c(\lambda) \int_{D \times D} f_1(z)\bar{f}_2(w)|1 - z\bar{w}|^{2\lambda} dxdydudv \tag{1.1}$$

where $z = x + iy, w = u + iv$, and

$$c(\lambda) = \frac{-\lambda - 1}{\pi}$$

Integral (1.1) converges absolutely for $\mathrm{Re}\lambda > -1$ and can be extended on the λ-plane by analyticity to a meromorphic function.

For $\lambda > -2$ the representation R_λ can be regarded as the tensor product of the analytic series representation T_σ^+ of the universal covering group $\tilde{G}$ of G and its conjugate representation T_σ^- where $\sigma = (-\lambda - 2)/2$.

Let $\langle \psi, \varphi \rangle$ be the inner product in $L^2(\bar{D})$ with respect to the Euclidean measure:

$$\langle \psi, \varphi \rangle = \int_D \psi(z)\bar{\varphi}(z) dxdy$$

It is invariant with respect to the pair $R_\lambda, R_{-\bar{\lambda}-2}$:

$$\langle R_\lambda(g)\psi, \varphi \rangle = \langle \psi, R_{-\bar{\lambda}-2}(g^{-1})\varphi \rangle \tag{1.2}$$

Extend R_λ to distributions on $\bar{D}$ by formula (1.2) where $\langle \psi, \varphi \rangle$ is understood as the value of $\psi \in \mathcal{D}'(\bar{D})$ at $\varphi \in \mathcal{D}(\bar{D})$.

Denote by Y the multiplication by

$$p = 1 - z\bar{z} \tag{1.3}$$

This operator intertwines R_λ and $R_{\lambda-1}$, so that Y^α intertwines R_λ and $R_{\lambda-\alpha}$:

$$p^\alpha R_\lambda(g) f = R_{\lambda-\alpha}(g)(p^\alpha f)$$

2 Poisson transform

Elementary representations T_σ, $\sigma \in \mathbb{C}$, (see, for example, [5]) of $G/\mathbb{Z}_2$ act on $\mathcal{D}(\Gamma)$ by the formula:

$$(T_\sigma(g)\varphi)(u) = \varphi(u \cdot g)|bu + \bar{a}|^{2\sigma},\ u \in \Gamma$$

The inner product from $L^2(\Gamma)$:

$$\langle \psi, \varphi \rangle_\Gamma = \int_0^{2\pi} \psi(u)\bar{\varphi}(u)d\alpha,\ u = e^{i\alpha}$$

is invariant with respect to the pair $(T_\sigma, T_{-\bar{\sigma}-1})$. The operator B_σ on $\mathcal{D}(\Gamma)$, defined by the formula

$$(B_\sigma\varphi)(u) = \int_0^{2\pi} |1 - u\bar{v}|^{-2\sigma-2}\varphi(v)d\beta,\ v = e^{i\beta}$$

intertwines T_σ and $T_{-\sigma-1}$. The basis $\psi_m(u) = e^{im\alpha}$, $m \in \mathbb{Z}$, consists of eigenfunctions of B_σ:

$$B_\sigma\psi_m = b_m(\sigma)\psi_m$$

where

$$b_m(\sigma) = 2\pi(-1)^m \frac{\Gamma(-2\sigma - 1)}{\Gamma(-\sigma + m)\Gamma(-\sigma - m)}.$$

For σ not integer T_σ and $T_{-\sigma-1}$ are equivalent (for σ integer there is a "partial equivalence").

For $\sigma \in \mathbb{C}$, define the Poisson transform $\mathcal{P}_\sigma$ by the formula

$$(\mathcal{P}_\sigma\varphi)(z) = \int_0^{2\pi} |1 - z\bar{u}|^{2\sigma}\varphi(u)d\alpha,\ u = e^{i\alpha}$$

It carries $\mathcal{D}(\Gamma)$ to $C^\infty(D)$ and intertwines $T_{-\sigma-1}$ and the restriction $R_{-\sigma-2}$ to $C^\infty(D)$:

$$R_{-\sigma-2}(g)\mathcal{P}_\sigma = \mathcal{P}_\sigma T_{-\sigma-1}(g)$$

It has the following asymptotics at the infinity, i.e. when $p \to 0$ (for p, see (1.3)):

$$(\mathcal{P}_\sigma\varphi)(z) \sim \sum_{k=0}^{\infty}(C_{\sigma,k}\varphi)(u) \cdot p^k + p^{2\sigma+1}\sum_{k=0}^{\infty}(D_{\sigma,k}\varphi)(u) \cdot p^k \tag{2.1}$$

where $z = ru, r = |z|, u = e^{i\alpha}$, and $C_{\sigma,k}$, $D_{\sigma,k}$ are some operators on $\mathcal{D}(\Gamma)$. There are connections between them:

$$C_{\sigma,k}B_\sigma = j(\sigma)D_{-\sigma-1,k} \tag{2.2}$$

$$D_{\sigma,k}B_\sigma = j(\sigma)C_{-\sigma-1,k} \tag{2.3}$$

where

$$j(\sigma) = b_0(\sigma) = 2\pi\Gamma(-2\sigma - 1)/\Gamma^2(-\sigma)$$

The basis functions ψ_m are transformed by $\mathcal{P}_\sigma$ to functions

$$(\mathcal{P}_\sigma\psi_m)(z) = \psi_m(u)(-1)^m 2\pi \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + m + 1)} p^\sigma P_\sigma^m\left(\frac{1 + r^2}{1 - r^2}\right) \tag{2.4}$$

where P_σ^m is the Legendre function, see [2], Ch 3. Applying [2] 3.2 (18), we express (2.4) in terms of the Gauss hypergeometric function:

$$(\mathcal{P}_\sigma\psi_m)(z) = \psi_m(u)(1 - p)^{-m/2}[b_m(-\sigma - 1)F(-\sigma, -\sigma - m; -2\sigma; p)$$

$$+p^{2\sigma+1}j(\sigma)F(\sigma + 1, \sigma + 1 - m; 2\sigma + 2; p)] \tag{2.5}$$

Comparing (2.5) with (2.1), we observe (because $(d^2/d\alpha^2)\psi_m = -m^2\psi_m$) that $D_{\sigma,k}$ are differential operators on Γ (and, by (2.2), (2.3), $C_{\sigma,k}$ are integral operators). Namely, consider the following power series in p:

$$(1 - p)^{-m/2}F(\sigma + 1, \sigma + 1 - m; 2\sigma + 2; p) \tag{2.6}$$

Its coefficients are polynomials in m with coefficients rational in σ. In virtue of [2] 2.1(23), the function (2.6) is invariant with respect to $m \mapsto -m$ so that it depends on m^2 only. Therefore, we have

$$(1-p)^{-m/2}F(\sigma+1,\sigma+1-m;2\sigma+2;p)=\sum_{k=0}^{\infty}W_k(\sigma,-m^2)p^k \qquad (2.7)$$

where $W_k(\sigma,t)$ are polynomials in t with coefficients rational in σ. Therefore, operators $D_{\sigma,k}$ are:

$$D_{\sigma,k}=j(\sigma)W_k(\sigma,\frac{d^2}{d\alpha^2}) \qquad (2.8)$$

Write a recurrence relation for W_k and several polynomials W_k:

$$(k+1)(2\sigma+k+2)W_{k+1}(\sigma,t)-[(\sigma+2k+1)^2-2k(k+1)]W_k(\sigma,t)+[(\sigma+k)^2+t/4]W_{k-1}(\sigma,t)=0$$

$$W_0(\sigma,t)=1$$

$$W_1(\sigma,t)=\frac{1}{2}(\sigma+1)$$

$$W_2(\sigma,t)=-\frac{1}{8(2\sigma+3)}t+\frac{(\sigma+1)(\sigma+2)^2}{4(2\sigma+3)}$$

$$W_3(\sigma,t)=-\frac{\sigma+3}{16(2\sigma+3)}t+\frac{(\sigma+1)(\sigma+2)(\sigma+3)^2}{24(2\sigma+3)}$$

$$W_4(\sigma,t)=\frac{1}{128(2\sigma+3)(2\sigma+5)}t^2-\frac{\sigma^3+10\sigma^2+33\sigma+35}{32(2\sigma+3)(2\sigma+5)}t+\frac{(\sigma+1)(\sigma+2)(\sigma+3)^2(\sigma+4)^2}{96(2\sigma+3)(2\sigma+5)}.$$

3 The diagonalization of representations on distributions concentrated at the boundary

Let us denote by $\Delta_m, m\in\mathbb{N}=\{0,1,2,\ldots\}$ the space of distributions on $\bar{D}$ having the form

$$\varphi(u)\delta^{(m)}(p)$$

where $\varphi\in\mathcal{D}(\Gamma), \delta^{(m)}$ the m-th derivative of the Dirac delta function; for p, see (1.3). The space

$$\Sigma_m=\Delta_0+\Delta_1+\ldots+\Delta_m$$

is invariant under R_λ (but each of $\Delta_1,\Delta_2,\ldots$ not)

Theorem 3.1 *Let $\lambda\notin 1/2+\mathbb{Z}$. Then for any $m\in\mathbb{N}$ there exists a unique subspace $V_m\subset\Sigma_m$ invariant and irreducible with respect to R_λ such that its projection to Δ_m is the whole Δ_m. The space Σ_m decomposes into the direct sum of irreducible invariant subspaces:*

$$\Sigma_m=V_0+V_1+\ldots+V_m \qquad (3.1)$$

The restriction of R_λ to V_m is equivalent to $T_{-\lambda-1+m}(\sim T_{\lambda-m})$. The decomposition (3.1) is orthogonal with respect to $(\,,\,)_\lambda$. A distribution ξ in V_m is characterized by its highest term $\varphi\cdot\delta^{(m)}(p)$, namely,

$$\xi=\sum_{j=0}^{m}(-1)^j\frac{m!}{(m-j)!}W_j(\lambda-m,\frac{d^2}{d\alpha^2})\varphi\cdot\delta^{(m-j)}(p) \qquad (3.2)$$

where W_j are polynomials defined by (2.7).

Proof. First let $m-1/2<\lambda<m+1/2, m\in\mathbb{N}$. Take an arbitrary function $\varphi\in\mathcal{D}(\Gamma)$ and denote

$$\Phi=\mathcal{P}_{\lambda-m}\varphi$$

The map $\varphi \mapsto p^m\Phi$ intertwines $T_{-\lambda-1+m}$ and $R_{-\lambda-2}$ and its image is irreducible subspace - where $R_{-\lambda-2}$ acts as $T_{-\lambda-1+m}(\sim T_{\lambda-m})$. The function $p^m\Phi$ has the following asymptotic when $p \to 0$ (see (2.1)),

$$(p^m\Phi)(z) \sim p^m \sum_{s=0}^{\infty} C_{\lambda-m,s}\varphi \cdot p^s + p^{2\lambda-m+1} \sum_{s=0}^{\infty} D_{\lambda-m,s}\varphi \cdot p^s \tag{3.3}$$

Apply to $p^m\Phi$ the operator $(\lambda-\mu)Y^{-2\mu-2}$ where μ is a complex parameter. This operator is the multiplication by $(\lambda-\mu)p^{-2\mu-2}$. The function $(\lambda-\mu)p^{-2\mu-2+m}\Phi$ is regarded as a distribution on $\bar{D}$. Let $\mu \to \lambda$. Then (see [3])

$$(\lambda-\mu)p^{2\lambda-2\mu-n-1} \to \frac{1}{2n!}(-1)^n\delta^{(n)}(p)$$

Therefore, by (3.3) our distribution tends to

$$\zeta = \frac{1}{2}\sum_{s=0}^{m} \frac{(-1)^{m-s}}{(m-s)!} D_{\lambda-m,s}\varphi \cdot \delta^{(m-s)}(p) \tag{3.4}$$

The map $\varphi \mapsto \zeta$ intertwines $T_{-\lambda-1+m}$ and R_λ, its image is exactly the desired subspace V_m. By analyticity we can free ourselves from the conditions $m-1/2 < \lambda < m+1/2, m \in \mathbb{N}$, so that $\lambda \notin 1/2+\mathbb{Z}$ remains only.

Substituting (2.8) in (3.4), we obtain (3.2). □

Write down distributions from V_m for $m = 0, 1, 2, 3, 4$:

$$\varphi\delta,$$

$$\varphi\delta' - \frac{\lambda}{2}\varphi\delta,$$

$$\varphi\delta'' - (\lambda-1)\varphi\delta' + \frac{1}{4(2\lambda-1)}[2\lambda^2(\lambda-1)\varphi - \varphi'']\delta,$$

$$\varphi\delta''' - \frac{3}{2}(\lambda-2)\varphi\delta'' + \frac{3}{4(2\lambda-3)}[2(\lambda-1)^2(\lambda-2)\varphi - \varphi'']\delta' +$$
$$+\frac{1}{8(2\lambda-3)}[-2\lambda^2(\lambda-1)(\lambda-2)\varphi + 3\lambda\varphi'']\delta,$$

$$\varphi\delta^{(4)} - 2(\lambda-3)\varphi\delta''' + \frac{3}{2(2\lambda-5)}[2(\lambda-2)^2(\lambda-3)\varphi - \varphi'']\delta'' +$$
$$+\frac{1}{2(2\lambda-5)}[-2(\lambda-1)^2(\lambda-2)(\lambda-3)\varphi + 3(\lambda-1)\varphi'']\delta' +$$
$$+\frac{1}{16(2\lambda-3)(2\lambda-5)}[4\lambda^2(\lambda-1)^2(\lambda-2)(\lambda-3)\varphi - 12(\lambda^3-2\lambda^2+\lambda-1)\varphi'' + 3\varphi^{(4)}]\delta.$$

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