A class of supercomplete systems of holomorphic functions

N. A. Malaschonok
Tambov State University, 392622 Tambov, Russia
e-mail: malaschonok@main.tsu.tambov.ru

Let $G \subset \mathbb{C}^p$ be a full p-circular domain, $\bar{G}$ – its closure, $\mathcal{A}(\bar{G})$ – the space of functions holomorphic in $\bar{G}$.

Denote $\delta_n(G) = \sup_{z \in G}|z^n|$ (we use the multi-index notation). Fix a sequence α_n of complex numbers such that

$$\lim_{|n|\to\infty}(\delta_n(G)|\alpha_n|^{-1})^{1/|n|} = 1$$

Then the function

$$h(z) = \sum_n \alpha_n^{-1} z^n$$

is holomorphic on G.

Let us take p sequences $\beta_k^{(1)}, \ldots, \beta_k^{(p)}$ of distinct complex numbers in unit disc ($|\beta_k^i| < 1$) with $|\beta_k^i| \to 1$ when $k \to \infty$. They give rise to the system of functions h_m:

$$h_m(z) = h(\beta_{m_1}^{(1)} z_1, \ldots, \beta_{m_1}^{(p)} z_p).$$

We call the system h_m *closed* in $\mathcal{A}(\bar{G})$, if for any function f in $\mathcal{A}(\bar{G})$ there exist numbers A_m such that

$$f(z) = \sum_m A_m h_m(z) \tag{1}$$

where the series converges absolutely and uniformly inside G (the convergence in $\mathcal{A}(\bar{G})$).

Remark. We have to put some conditions on $\beta_k^{(i)}$ in order h_m be closed.

Let us call a system h_m *supercomplete*, if there exists a nontrivial representation of zero with respect to this system.

Theorem *A system h_m is supercomplete if and only if it is closed.*

Proof. Firstly, let h_m be closed. Let us take an arbitrary function f in $\mathcal{A}(\bar{G})$

$$f(z) = \sum_n a_n z^n. \tag{2}$$

There exist A_m such that (1) holds. Define the operator D on $\mathcal{A}(\bar{G})$:

$$(Df)(z) = \sum_{n=0}^{\infty} a_{n+1} \frac{\alpha_n}{\alpha_{n+1}} z^n.$$

It is linear and continuous in $\mathcal{A}(\bar{G})$. Let us take p complex numbers γ_i, $|\gamma_i| < 1, \gamma_i \neq \beta_k^{(i)}$, $i = 1, \ldots, p$; $k = 1, 2, \ldots$. The function $h_\gamma(z) = h(\gamma z)$ is an eigenfunction of D:

$$(Dh_\gamma)(z) = (\prod_{i=1}^{p} \gamma_i) h_\gamma(z).$$

Being in $\mathcal{A}(\bar{G})$, it is represented according to (1):

$$h_\gamma(z) = \sum_m A_{\gamma m} h_m(z). \tag{3}$$

Applying D to (3), we obtain:

$$(\prod_{i=1}^{p}\gamma_i)h_\gamma(z)=\sum_m A_{\gamma m}\prod_{i=1}^{p}\beta_m^{(i)}h_m(z). \quad (4)$$

Multiplying (3) by $\prod_{i=1}^{p}\gamma_i$, and subtracting from (4), we obtain a nontrivial representation of zero in $A(\bar{G})$:

$$0=\sum_m A_{\gamma m}(\prod_{i=1}^{p}\beta_m^{(i)}-\prod_{i=1}^{p}\gamma_i)h_m(z).$$

The second part of the proof includes three lemmae. For f of form (2) define

$$\tilde{f}(z)=\sum_n a_n\alpha_n z^n$$

. Let I denote the unit polydisk $|z_1|\leq 1,\ldots,|z_p|\leq 1$.

Lemma 1. ([2]) $f\in\mathcal{A}(\bar{G})$ *if and only if* $\tilde{f}\in\mathcal{A}(\bar{I})$.

Lemma 2. *The function f admits the representation (1) in $\mathcal{A}(\bar{G})$ if and only if*

$$\tilde{f}(z)=\sum_m A_m\prod_{i=1}^{p}(1-\beta_{m_i}^{(i)}z)^{-1}$$

in $\mathcal{A}(\bar{I})$.

Let h_m be supercomplete, i.e. there exist B_m ($B_m\neq 0$ for the infinite set of m) such that $0=\sum_m B_m h_m(z)$ in $\mathcal{A}(\bar{G})$. By Lemma 2 we have

$$0=\sum_m B_m\prod_{i=1}^{p}(1-\beta_{m_i}^{i}z_i)^{-1}$$

in $\mathcal{A}(\bar{I})$. Fix i and sum B_m over all indexes m_j with $j\neq i$. We obtain the number $d(i,m_i)$. Let λ be a number such that $|\lambda|<1,\lambda\neq\beta_k^i$. Consider the functions

$$g_i(\lambda,\xi)=\sum_{k=1}^{\infty}d(i,k)/(\beta_k^{(i)}-\lambda)(1-\beta_k^{(i)}\xi),\ i=1,\ldots,p.$$

Lemma 3.*(cf [1]). For any $z\in G$ we have:*

$$1/(1-\lambda z)=(1/g_i(\lambda,0))\sum_{k=1}^{\infty}d(i,k)/(\beta_k^{(i)}-\lambda)(1-\beta_k^{(i)}\xi).$$

($g_i(\lambda,0)\neq 0$ holds).

Let us take $f\in\mathcal{A}(\bar{G})$ and the associated $\tilde{f}\in\mathcal{A}(\bar{I})$. In $\mathbb{C}$, consider the torus $T(R)$ "of radius" $R>1$: $|t_i|=R,\ i=1,\ldots,p$. Take R such that f is holomorphic in RG and $|g_i(1/t_i,0)|\geq d>0$ for $|t_i|=R$. Then by the Cauchy formula we have

$$\tilde{f}(z)=\frac{1}{(2\pi i)^p}\int_{T(R)}\tilde{f}(t)\prod_{i=1}^{p}\frac{dt_i}{t_i-z_i}.$$

Applying Lemma 3 we obtain:

$$\tilde{f}(z)=\frac{1}{(2\pi i)^p}\int_{T(R)}\tilde{f}(t)\prod_{i=1}^{p}\frac{dt_i}{t_i(1-z_i/t_i)}$$

$$=\frac{1}{(2\pi i)^p}\int_{T(R)}\tilde{f}(t)\prod_{i=1}^{p}\frac{dt_i}{t_ig_i(1/t_i,0)}\prod_1^p\sum_m\frac{d_(i,m_i)}{(\beta_{m_i}^{(i)}-1/t_i)(1-\beta_{m_i}^{(i)}z_i)}$$

$$\sum_m\frac{A_m}{\prod_{i=1}^{p}(1-\beta_{m_i}^{(i)}z_i)},$$

where

$$A_m = \prod_1^p d_(i, m_i) \frac{1}{(2\pi i)^p} \int_{T(R)} \tilde{f}(t) \prod_{i=1}^p \frac{dt_i}{t_i g_i(1/t_i, 0)(1 - \beta_{m_i}^{(i)} t_i)}.$$

So by Lemma 2, we have in $\mathcal{A}(\bar{G})$:

$$f(z) = \sum_m A_m h_m(z)$$

with A_m defined by (4) □

In conclusion let us give examples of supercomplete systems, where the sequence α_n, and therefore the function $h(z)$, appears in a reasonable way.

Let $G^{(2)}$ be the "square" of G, i.e. $G^{(2)}$ consists of $(z_1^2, \ldots, z_p^2)$ for $z \in G$. Take the Szegö kernel $\Psi(z, \bar{\xi}) = \sum \sigma_n z^n \bar{\xi}^n$ of $G^{(2)}$. Then [2] we may take $\alpha_n = \sigma_n^{-1}$. In particular, using [2], we can write $h(z)$ for domains $|z_1|^{1/l} + |z_2| < 1$ and $|z_1|^{1/2} + |z_2|^{1/2} < 1$, respectively:

$$h(z) = (1 - z_2)^{l-1}[(1 - z_2)^l - z_1]^{-2},$$

$$h(z) = \frac{(1 - z_1 - z_2)^2(1 + z_1 + z_2) + 8z_1 z_2}{[(1 - z_2)^2 - 4z_1 z_2]^2}$$

We hope systems above can be used for quantization.

References

1 Yu.F.Korobeinik, A contribution to the representation of holomorphic functions by the series in rational functions, Matem. Zametki, 1982, t.31, N5, 3-16.

2 L.A.Aizenberg, Continuation of integral representations with the kernels and quasi-kernels Szegö for n-circular domains, Krasnoyarsk, Preprint IF SO AN SSSR, 1973, 3-34.