Maximal degenerate series for SL(n, R) of rank two

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We consider representations of the group SL($n,\mathbb{R}$) induced by characters of standard maximal parabolic subgroups $P^{\pm}$ corresponding to the partition $n=(n-2)+2$. These representations can be realized on functions on the Grassmann manifolds of rank two. It explains the title. In this paper we restrict ourselves to the structure of these representations (irreducibility and reducibility). Other properties (intertwining operators, invariant Hermitian forms, unitarizability) will be considered elsewhere.

1 Maximal degenerate series of SL(n, R) of rank two

Let $G = \mathrm{SL}(n,\mathbb{R}), n \geq 4$. Any element $g \in G$ can be written as a block matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.1}$$

according to the partition $n = (n-2)+2$ of n. Let $P^{\pm}$ denote the two maximal parabolic subgroups of G corresponding to this partition. They consist of upper and lower block matrices respectively:

$$P^+ : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, P^- : \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \tag{1.2}$$

For $\mu \in \mathbb{C},\ \varepsilon = 0,1,$ let us denote by $t^{\mu,\varepsilon}$ the character of R^*:

$$t^{\mu,\varepsilon} = |t|^{\mu}\,\mathrm{sgn}^{\varepsilon} t$$

Define the character $\omega_{\mu,\varepsilon}$ of $P^{\pm}$ by formula:

$$\omega_{\mu,\varepsilon}(p) = (\det c)^{\mu,\varepsilon}$$

where p has one of forms (1.2). Consider the representations

$$T^{\pm}_{\mu,\varepsilon} = \mathrm{Ind}(G, P^{\mp}, \omega_{\mp\mu,\varepsilon})$$

Let us describe these representations in "the compact picture". Let $K = \mathrm{SO}(n)$, a maximal compact subgroup of G . One has the following decompositions

$$G = P^+K = P^-K \tag{1.3}$$

(Iwasawa type decompositions). For the corresponding decompositions $g = pk$ the factors p and k are defined up to the factor in $L = K \cap P^+ = K \cap P^- : pk = p_1k_1$ with $p_1 = pl^{-1}$, $k_1 = lk, l \in L$. The subgroup L is $\mathrm{S}(\mathrm{O}(n-2) \times \mathrm{O}(2))$.

Take the Iwasawa type decomposition $g = pk, p \in P^+, k \in K$, with g and p given by (1.1) and (1.2) respectively. For the block c we have the equation $cc' = \gamma\gamma' + \delta\delta'$ (the prime denotes matrix transposition). We can take for c a symmetric positive definite matrix:

$$c = (\gamma\gamma' + \delta\delta')^{1/2} \tag{1.4}$$

The connected component L_e of L containing the identity element is $\mathrm{SO}(n-2)\times\mathrm{SO}(2)$. The coset space $S = K/L_e$ is Grassmann manifold of rank two, the manifold of two-dimensional oriented planes in

R^n. We let K act on S from the right (i.e. S is the right coset space), so that S can be realized as the space of real $2 \times n$ matrices

$$s = \begin{pmatrix} u_1 & \dots & u_n \\ v_1 & \dots & v_n \end{pmatrix}$$

with the condition $ss' = E$ (so that the rows of s are unit orthogonal vectors) and the identification $s \sim rs, r \in \mathrm{SO}(2)$. We shall denote by s the equivalence class containing s. Let us denote

$$s^0 = \begin{pmatrix} 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Let $\mathcal{D}_\varepsilon(S)$ be the space of complex valued C^∞ functions on S of parity ε :

$$\varphi(ws) = (\det w)^\varepsilon \varphi(s), w \in \mathrm{O}(2)$$

(we write functions on S as $\varphi(s)$ keeping in mind the agreement above or thinking of φ to be left invariant with respect to SO(2)).

The group G acts on S (from the right):

$$s \longmapsto \widetilde{s} = s \cdot g \tag{1.5}$$

as follows. Let k be an element of K such that $s^0 k = s$. Decompose kg according to (1.3): $kg = \widetilde{p}\widetilde{k}, \widetilde{p} \in P^+, \widetilde{k} \in K$. Then $\widetilde{s} = s^0 \widetilde{k}$. The restriction of this action to K is the action by translations: $s \cdot k = sk$.

The representation $T^-_{\mu,\varepsilon}$ acts on $\mathcal{D}_\varepsilon(S)$ by the formula:

$$\left(T^-_{\mu,\varepsilon}(g)\varphi\right)(s) = \varphi(\widetilde{s})\,(\det \widetilde{c})^{\mu,\varepsilon} \tag{1.6}$$

where $\widetilde{c}$ is the c-block of the matrix $\widetilde{p}$. The definitions (1.5) and (1.6) are well-defined.

Let us write (1.6) in a more detailed way. Using the choice (1.4), we obtain

$$\widetilde{c} = (sgg's')^{1/2}.$$

Therefore

$$\left(T^-_{\mu,\varepsilon}(g)\varphi\right)(s) = \varphi\left((sgg's')^{-1/2}sg\right)\{\det(sgg's')\}^{\mu/2}$$

The representation T^+ is reduced to T^- by the outer automorphism $g \longmapsto g'^{-1}$(a Cartan involution):

$$T^+_{\mu,\varepsilon}(g) = T^-_{\mu,\varepsilon}(g'^{-1})$$

The representations $T^\pm_{\mu,\varepsilon}$ are continuous [1], Ch.8, in the sense that the function $T^\pm_{\mu,\varepsilon}(g)\varphi$ from $G \times \mathcal{D}_\varepsilon(S)$ into $\mathcal{D}_\varepsilon(S)$ in continuous, and indefinitely differentiable, i.e. for each $\varphi \in \mathcal{D}_\varepsilon(S)$ the function $T^\pm_{\mu,\varepsilon}(g)\varphi$ from G to $\mathcal{D}_\varepsilon(S)$ is of class C^∞. We preserve the symbols of these representations for the corresponding representations of the Lie algebra $\mathfrak{g}$ of G .

For $n = 4$ one has a homomorphism of G onto $SO_0(3,3)$ with the kernel $\{\pm E\}$. On the other hand, under $T^\pm_{\mu,\varepsilon}$ this kernel goes to the indentity operator. So in fact $T^\pm_{\mu,\varepsilon}$ are representations of $SO_0(3,3)$. They were studied in [4]. So from now on we shall assume $n > 4$. Notice that for $n = 4$ the subgroups P^+ and P^- are conjugated by an inner automorphism – the conjugation by means of the matrix

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

so that $T^+_{\mu,\varepsilon}$ and $T^-_{\mu,\varepsilon}$ are equivalent.

2 Harmonic analysis on the Grassmann manifold

The manifold $S = K/L_e$ is a compact symmetric space, indeed, L is the fixed point subgroup of the involution $\sigma(k) = IkI$ where I is the diagonal block matrix with the diagonal $1, \dots, 1-1, -1$. The Lie algebra $\mathfrak{k}$ of K is the direct sum $\mathfrak{k}=\mathfrak{l}+\mathfrak{m}$ of eigenspaces of σ. The rank of S is equal to 2. The quasiregular representations π of K on $\mathcal{D}(S)$ (the representations by translations) is the direct sum of the representations $\pi^{(\varepsilon)}$, $\varepsilon = 0, 1$, acting on $\mathcal{D}_\varepsilon(S)$. Since S is a symmetric space, both $\pi^{(\varepsilon)}$ decompose into the multiplicity free direct sum of irreducible representations. Each irreducible subspace contains exactly one (up to the factor) spherical function, it is constant on L_e-orbits. So let us describe these orbits. Let us take in $\mathfrak{m}$ the maximal Abelian subalgebra $\mathfrak{a}$ consisting of matrices

$$X = t_1(E_{n1} - E_{1n}) + t_2(E_{n-1,2} - E_{2,n-1})$$

where E_{ij} is the standard matrix basic. Let $A = \exp \mathfrak{a}$. The set $s^0 A$ consists of points

$$s = \begin{pmatrix} 0 & \sin t_2 & 0 & \dots & 0 & \cos t_2 & 0 \\ \sin t_1 & 0 & 0 & \dots & 0 & 0 & \cos t_1 \end{pmatrix}$$

One has the Cartan decomposition $K = L_e A L_e$. Hence any L_e-orbit on S is completely defined by its intersection with $s^0 A$. This intersection is obtained from a given point by the following transformations: a) $t_1 \to -t_1$, b) $t_2 \to -t_2$, c) $(t_1, t_2) \to (t_1 + \pi, t_2 + \pi)$, d) $(t_1, t_2) \to (t_2, t_1)$. Therefore, L_e-orbits can be parameterized by the two functions x, y:

$$x = (1/2)(\cos^2 t_1 + \cos^2 t_2), \quad y = \cos t_1 \cos t_2, \tag{2.1}$$

so that functions φ on S constant on L_e-orbits are functions of x, y:

$$\varphi(s) = F(x, y) \tag{2.2}$$

For variables x, y given by (2.1), points (x, y) fill out the domain D on the plane xOy defined by inequalities $|y| \leq x \leq (1/2)(y^2 + 1)$. Let ds be a K-invariant measure on S. For functions φ invariant with respect to L_e (i.e. of the form (2.2)) this measure gives rise to the measure

$$\begin{aligned} C|\sin t_1 \sin t_2|^{n-4}|\sin^2 t_1 - \sin^2 t_2| dt_1 dt_2 = \\ = C(1 - 2x + y^2)^{(n-5)/2} dx dy \end{aligned} \tag{2.3}$$

Let $\langle\, , \rangle$ denote the inner product correspoding to the measure (2.3) (with $C = 1$):

$$\langle F_1, F_2 \rangle = \int_D F_1(x, y)\overline{F_2(x, y)}(1 - 2x + y^2)^{(n-5)/2}\, dx\, dy \tag{2.4}$$

Similarly, L-orbits on S can be parameterized by the functions (they were used in [3]):

$$\xi = \sin^2 t_1 + \sin^2 t_2 = 2(1 - x), \; \eta = \sin^2 t_1 \sin^2 t_2 = 1 - 2x + y^2 \tag{2.5}$$

In [3], Koornwinder introduced a family of polynomials $R_{l,m}^{\alpha,\beta,\gamma}(\xi, \eta)$ in ξ, η with highest term $const \cdot \xi^{l-m}\eta^m$ which are obtained by orthogonalization of the sequence $1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \dots$ with respect to the measure

$$\eta^\alpha (1 - \xi + \eta)^\beta (\xi^2 - 4\eta)^\gamma d\xi d\eta$$

on the domain $2\sqrt{\eta} \leq \xi \leq 1 + \eta$.

Theorem 2.1. *Irreducible constituents ${\pi_z}^{(\varepsilon)}$ of $\pi^{(\varepsilon)}$ can be labelled by pairs $z = (l, m)$ of integers with $l \geq m \geq 0$. The spherical function ${\Phi_z}^{(\varepsilon)}$ in the corresponding space ${V_z}^{(\varepsilon)} \subset \mathcal{D}_\varepsilon(S)$ normalized by ${\Phi_z}^{(\varepsilon)}(s_0)=1$ is the following polynomial in x, y, see (2.1) and (2.2):*

$${\Phi_z}^{(\varepsilon)}(x, y) = y^\varepsilon R_{l,m}^{\alpha,\beta,\gamma}(\xi, \eta) \tag{2.6}$$

with $\alpha = (n-5)/2, \beta = \varepsilon - 1/2, \gamma = 0$ where ξ, η are expressed in terms of x, y by (2.5).

Proof. For ε=0 the theorem was proved in [3]. For ε =1 the theorem is proved by the direct check that functions (2.6) satisfy differential equations defining spherical functions □.

Write down explicit expressions. We have

$$\Phi_z{}^{(\varepsilon)} = \sum d^{(\varepsilon)}_{l,m;k,r} y^\varepsilon W_{k,r}(x,y) \tag{2.7}$$

where the summation is taken over (k,r) such that $0 \le r \le k \le l$, $r \le m$,

$$d^{(\varepsilon)}_{l,m;k,r} = (2k-2r+1)\frac{(-m)^{[r]}(-l)^{[k]}(-l-\frac{1}{2})^{[r]}(l+\varepsilon+\frac{n-3}{2})^{[k]}(m+\varepsilon+\frac{n-4}{2})^{[r]}}{(-l)^{[r]}(\frac{n-2}{2})^{[k]}(\frac{n-3}{2})^{[r]}(\frac{3}{2})^{[k]}r!}\cdot$$

$$\cdot {}_4F_3\left(\begin{matrix}-k+r,-l+m,-l-m-\varepsilon-(n-4)/2,1/2; \\ -l+r,-l-k-\varepsilon-(n-5)/2,1;\end{matrix}\ 1\right),$$

$$W_{k,r}(x,y) = (1-2x+y^2)^{(k+r)/2}P_{k-r}(\frac{1-x}{\sqrt{1-2x+y^2}}),$$

$P_j(t)$ is the Legendre polynomial, $a^{[k]} = a(a+1)\dots(a+k-1)$. One can see that $\Phi_z{}^{(\varepsilon)}$ is a linear combination of functions $(1-x)^{p-q}(1-2x+y^2)^q y^\varepsilon$ where $p \le l, p+q \le l+m$.

3 The structure of maximal degenerate series representations

Let (,) be the inner product in $L^2(S,ds)$:

$$(\varphi,\psi) = \int\limits_S \varphi(s)\overline{\psi(s)}ds \tag{3.1}$$

The measure ds is transformed under (1.11) as follows:

$$d\widetilde{s} = |\det \widetilde{c}|^{-n}\, ds$$

Therefore the form (3.1) is invariant with respect to the pairs $\left(T^+_{\mu,\varepsilon}, T^+_{-\overline{\mu}-n,\varepsilon}\right)$ and $\left(T^-_{\mu,\varepsilon}, T^-_{-\overline{\mu}-n,\varepsilon}\right)$:

$$\left(T^\pm_{\mu,\varepsilon}(g)\varphi,\psi\right) = \left(\varphi, T^\pm_{-\overline{\mu}-n}(g^{-1})\psi\right) \tag{3.2}$$

So $T^\pm_{\mu,\varepsilon}$ are unitarizable for $\mathrm{Re}\mu = -n/2$, the invariant inner product is (3.1). We shall see below, that these representations are irreducible. Their unitary completions form the *continuous series* of unitary irreducible representations.

The centralizer of L in g is one-dimensional, let us take for a basis the element

$$Z_0 = \mathrm{diag}\left\{\frac{2}{n},\dots,\frac{2}{n},\frac{2}{n}-1,\frac{2}{n}-1\right\}$$

The operator $T^\pm_{\mu,\varepsilon}(Z_0)$ preserves the set of L_e-invariant functions in $\mathcal{D}_\varepsilon(S)$, so that it gives rise to a differential operator in variables x,y . This differential operator is $\pm 2\mathcal{L}_\mu$ where

$$\mathcal{L}_\mu = \left(2x^2-x-y^2\right)\frac{\partial}{\partial x} + (x-1)\,y\frac{\partial}{\partial y} + \mu\left(\frac{2}{n}-x\right)$$

It follows from (3.2) that

$$\langle \mathcal{L}_\mu F_1, F_2\rangle = \langle F_1, \mathcal{L}_{-\overline{\mu}-n}F_2\rangle$$

Let us take 4 vectors on the plane:

$$e_1 = (1,0), e_2 = (0,1), e_3 = (0,-1), e_4 = (-1,0)$$

and for $\mu \in \mathbb{C}, \varepsilon = 0,1$ define 4 linear functions of $z = (l,m)$:

$$\beta_1(\mu,\varepsilon;z) = (1/2)(\mu-\varepsilon) - l,$$

$$\beta_2(\mu,\varepsilon;z) = (1/2)(\mu+1-\varepsilon) - m,$$

$$\beta_3(\mu,\varepsilon;z) = (1/2)(\mu+n-3+\varepsilon)+m,$$

$$\beta_4(\mu,\varepsilon;z) = (1/2)(\mu+n-2+\varepsilon)+l.$$

There are relations between them:

$$\beta_i(\mu,\varepsilon;z)+\beta_{5-i}(\mu,\varepsilon;z) = \mu+(n-2)/2$$

$$\beta_i(\mu,\varepsilon;z)+\beta_{5-i}(\mu^*,\varepsilon;z) = -1$$

where

$$\mu^* = -\mu-n$$

Lemma 3.1 *The operator $\mathcal{L}_\mu$ carries $W_z = W_{l,m}$ to a linear combination of $W_z, W_{z+e_1}, W_{z+e_2}$:*

$$\mathcal{L}_\mu W_z = \left(-\tfrac{n-2}{2}\mu+l+m\right)W_z+\left(\tfrac{\mu}{2}-l\right)W_{z+e_1}+ \\ +(l-m)^2\left[(l-m)^2-\tfrac{1}{4}\right]^{-1}\left(\tfrac{\mu+1}{2}-m\right)W_{z+e_2}$$

The lemma is proved by means of the differentiation formulae for the Legendre polynomials, see, for example, [2] 10.10.

Lemma 3.2 *The operator $\mathcal{L}_\mu$ carries $\Phi_z^{(\varepsilon)}$ to a linear combination of $\Phi_z^{(\varepsilon)}$ and neighboring functions $\Phi_{z+e_i}^{(\varepsilon)}$:*

$$\mathcal{L}_\mu\Phi_z^{(\varepsilon)} = \gamma_0(\mu,\varepsilon;z)\Phi_z^{(\varepsilon)}+\sum_{i=1}^{4}\gamma_i(\varepsilon;z)\beta_i(\mu,\varepsilon;z)\Phi_{z+e_i}^{(\varepsilon)}$$

where

$$\gamma_1 = -\frac{(l-m+1)\left(l+\frac{n-2}{2}\right)\left(l+\frac{n-3}{2}+\varepsilon\right)\left(l+m+\frac{n-2}{2}+\varepsilon\right)}{\left(l-m+\frac{1}{2}\right)\left(l+m+\frac{n-3}{2}+\varepsilon\right)\left(2l+\frac{n}{2}+\varepsilon\right)\left(2l+\frac{n-2}{2}+\varepsilon\right)}$$

$$\gamma_2 = -\frac{(l-m)\left(m+\frac{n-3}{2}\right)\left(m+\frac{n-4}{2}+\varepsilon\right)\left(l+m+\frac{n-2}{2}+\varepsilon\right)}{\left(l-m+\frac{1}{2}\right)\left(l+m+\frac{n-3}{2}+\varepsilon\right)\left(2m+\frac{n-2}{2}+\varepsilon\right)\left(2m+\frac{n-4}{2}+\varepsilon\right)}$$

$$\gamma_3 = -\frac{(l-m+1)\,m\left(m-\frac{1}{2}+\varepsilon\right)\left(l+m+\frac{n-4}{2}+\varepsilon\right)}{\left(l-m+\frac{1}{2}\right)\left(l+m+\frac{n-3}{2}+\varepsilon\right)\left(2m+\frac{n-6}{2}+\varepsilon\right)\left(2m+\frac{n-4}{2}+\varepsilon\right)}$$

$$\gamma_4 = -\frac{(l-m)\left(l+\frac{1}{2}\right)(l+\varepsilon)\left(l+m+\frac{n-4}{2}+\varepsilon\right)}{\left(l-m+\frac{1}{2}\right)\left(l+m+\frac{n-3}{2}+\varepsilon\right)\left(2l+\frac{n-4}{2}+\varepsilon\right)\left(2l+\frac{n-2}{2}+\varepsilon\right)}$$

$$\gamma_0 = (n-4)\left(\tfrac{\mu}{2}+\tfrac{n}{4}\right)\cdot$$

$$\cdot\left[-\frac{1}{n}+\frac{n-6}{2}\cdot\frac{l^2+m^2+\left(\frac{n-2}{2}+\varepsilon\right)l+\left(\frac{n-4}{2}+\varepsilon\right)m+\frac{1}{2}\left(\frac{n-2}{2}+\varepsilon\right)\left(\frac{n-4}{2}+\varepsilon\right)}{\left(2l+\frac{n-4}{2}+\varepsilon\right)\left(2l+\frac{n}{2}+\varepsilon\right)\left(2m+\frac{n-2}{2}+\varepsilon\right)\left(2m+\frac{n-6}{2}+\varepsilon\right)}\right]$$

The lemma is proved using Lemma 3.1, formulae (2.7), (3.3) and pairwise orthogonality of $\Phi_z^{(\varepsilon)}$ with respect to the inner product (2.4) and some formulae about ${}_4F_3$ from [6]. We omit these rather cumbersome calculations. In particular, in certain cases we resorted to the help of a computer (Maple V).

Let Z be the lattice of the pairs $z=(l,m)$ of integers with $l\geq m\geq 0$ (K-types for $\pi^{(\varepsilon)}$). Let us call the line $\beta_i(\mu,\varepsilon;z)=0$ on the z-plane a barrier if it meets the lattice Z and the intersection $Z\cap\{\beta_i\geq 0\}$ does not coincide with Z. Then μ has to be integer. If the line $\beta_i=0$ is a barrier, we denote by $V_i(\mu,\varepsilon)$ the subspace given by the inequality $\beta_i(\mu,\varepsilon;z)\geq 0$, i.e. the sum of $\pi_z^{(\varepsilon)}$ with z satisfying this inequality. The following lemma follows from the definition of β_i immediately.

Lemma 3.3 *For given μ,ε the number of barriers is not greater than 1.*

Let us point out μ,ε ($\mu\in\mathbb{Z}$) for which $\beta_i=0$ is a barrier (the sign $\equiv$ denotes the congruence modulo 2):

$i=1:\mu\geq\varepsilon,\mu\equiv\varepsilon$
$i=2:\mu\geq\varepsilon-1,\mu\equiv\varepsilon-1$
$i=3:\mu\leq 1-n-\varepsilon,\mu\equiv 1-n-\varepsilon$ (i.e. $\mu^*\geq\varepsilon-1,\mu^*\equiv\varepsilon-1$)
$i=4:\mu\leq -n-\varepsilon,\mu\equiv n+\varepsilon$ (i.e. $\mu^*\geq\varepsilon,\mu^*\equiv\varepsilon$)

Theorem 3.4 *If the line $\beta_i(\mu,\varepsilon;z)=0$ is a barrier, then the subspace $V_i(\mu,\varepsilon)$ is invariant for $T^{\pm}_{\mu,\varepsilon}$. There is no other invariant subspaces. The representation $T^{\pm}_{\mu,\varepsilon}$ is irreducible except when μ is integer and satisfies one of two inequalities: $\mu \geq \varepsilon - 1$ or $\mu \leq 1-n-\varepsilon$ (i.e. $\mu^* \geq \varepsilon - 1$). In this case there is exactly one invariant subspace $V_i(\mu,\varepsilon)$(see the list after Lemma 3.3).*

The proof is carried out analogously to the proofs of similar theorems in [4], [5].

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