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Harald Bohr's theorem for bounded symmetric domains

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For a holomorphic function from the unit disc into itself, the well known Harald Bohr's theorem estimates the sum of absolute values of its Taylor components. In the paper, this result is generalized to bounded circled symmetric domains

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§ 0. Introduction

0.1. The following theorem was proved by Harald Bohr [1] for $|z| < 1/6$, then soon improved to $|z| < 1/3$ from remarks by M. Riesz, I. Schur, F. Wiener [1]:

Theorem 1 (Bohr 1914) *Let $\Delta \subset \mathbb{C}$ be the unit disc $|z| < 1$. Let $f : \Delta \to \Delta$ be a holomorphic function from the disc into itself with the Taylor expansion*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots .$$

Then

$$\sum_{k=0}^{\infty} \left| a_k z^k \right| < 1 \tag{1}$$

for $|z| < 1/3$. The value $1/3$ is optimal.

This result has been extended by Liu Taishun and Wang Jianfei ([3], 2007) to the bounded symmetric domains of the four classical series, and to polydiscs, using a case-by-case analysis, as follows:

Theorem 2 *Let Ω be an irreducible bounded symmetric domain of classical type in the sense of Hua Luokeng [4], or a polydisc. Denote by $\| \|_\Omega$ the Minkowski norm associated to Ω. Let $f : \Omega \to \Omega$ be a holomorphic map and let*

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

[1] See [2] for biographical elements about Friedrich Wiener.

be its Taylor expansion in k-homogeneous polynomials f_k. Let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(f(0)) = 0$. Then

$$\sum_{k=0}^{\infty} \frac{\|D\phi(f(0)) \cdot f_k(z)\|_\Omega}{\|D\phi(f(0))\|_\Omega} < 1 \tag{2}$$

for all z such that $\|z\|_\Omega < 1/3$.

For $\|z\|_\Omega > 1/3$, there exists a holomorphic map $f : \Omega \to \Omega$ such that (2) is not true.

In [5], we give a classification independent proof of this result, which is valid for any bounded circled symmetric domain.

0.2. In the above generalization of Bohr's theorem, one considers the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

of a bounded holomorphic function or map, and one asks for which z the inequality (2) holds. One may ask the same type of question for other decompositions of f. For example, the following problem has been considered recently by several authors:

Problem 1. Let Ω be the unit ball of the ℓ^p norm:

$$\Omega = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} |z_k|^p < 1 \right\}.$$

Let $f : \Omega \to \Delta$ be a holomorphic function and consider the expansion of f in monomials

$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

Determine the best constant K such that $z \in K\Omega$ ensures

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \left| a_{k_1\cdots k_n} z_1^{k_1} \ldots z_n^{k_n} \right| < 1$$

for all $f : \Omega \to \Delta$.

For results about this type of problem, see [6], [7], [8], [9], [10], [11].

§ 1. H. Bohr's theorem

1.1. We present here a rather elementary proof of Bohr's theorem (Theorem 1), based on a lemma of F. Wiener. There are other proofs of Bohr's theorem in the literature, for example by S. Sidon ([12], 1927), following L. Fejér's method of positive kernels [13]; the same proof was published later by M. Tomić ([14], 1962).

For a holomorphic function $f : \Delta \to \Delta$ with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, denote by $\mathfrak{M}(f, r)$ the sum of absolute values:

$$\mathfrak{M}(f, r) = \sum_{k=0}^{\infty} |a_k| \, r^k.$$

Lemma 1 (F. Wiener) *Let $f : \Delta \to \Delta$ be a holomorphic function, with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then for all $k > 0$,*

$$|a_k| \leqslant 1 - |a_0|^2 . \tag{3}$$

See [15] for an elementary proof of this lemma.

Proof of Bohr's theorem. (a) Direct part. Using (3), we deduce

$$\mathfrak{M}(f, r) = \sum_{k=0}^{\infty} |a_k| \, r^k \leqslant |a_0| + (1 - a_0 \overline{a_0}) \, \frac{r}{1 - r}$$

and, for $r \leqslant 1/3$,

$$\mathfrak{M}(f, r) \leqslant |a_0| + \frac{1}{2} \, (1 - a_0 \overline{a_0}) = 1 - \frac{1}{2} \left(1 - |a_0|^2\right) < 1,$$

as $|a_0| = |f(0)| < 1$.

(b). To prove that the bound 1/3 in Theorem 1 cannot be improved, one considers, for $0 < \alpha < 1$, the function

$$f(z) = \frac{\alpha - z}{1 - \alpha z}$$

and one checks easily that $\mathfrak{M}(r) > 1$ when $r > (1 + 2\alpha)^{-1}$; as $(1 + 2\alpha)^{-1} \to 1/3 + 0$ when $\alpha \to 1 - 0$, there exists for each $r > 1/3$ a holomorphic function $f : \Delta \to \Delta$ such that $\mathfrak{M}(f, r) > 1$. □

1.2. Generalizations in dimension 1. Other Bohr type theorems hold in dimension 1 for various classes of holomorphic functions $f : \Delta \to \Delta$. Consider the following classes of analytic functions on the unit disc:

$$\mathcal{F}_m = \left\{ f : \Delta \to \Delta \mid f(z) = \sum_{k=m}^{\infty} a_k z^k \right\} \qquad (m \in \mathbb{N}),$$

$$\mathcal{F}_{m,\alpha} = \left\{ f : \Delta \to \Delta \mid f(z) = \alpha z^m + \sum_{k=m+1}^{\infty} a_k z^k \right\} \qquad (m \in \mathbb{N}, \quad 0 \leqslant \alpha < 1).$$

Define the bounding functions

$$\mathfrak{M}_m(r) = \sup\{\mathfrak{M}(f,r) \mid f \in \mathcal{F}_m\} \qquad (m \in \mathbb{N}),$$
$$\mathfrak{M}_{m,\alpha}(r) = \sup\{\mathfrak{M}(f,r) \mid f \in \mathcal{F}_{m,\alpha}\} \qquad (m \in \mathbb{N}, \quad 0 \leqslant \alpha < 1)$$

and the corresponding *Bohr numbers* by

$$B_m = \sup\{r \mid \mathfrak{M}_m(r) < 1\},$$
$$B_{m,\alpha} = \sup\{r \mid \mathfrak{M}_{m,\alpha}(r) < 1\}.$$

With these notations, the original Bohr theorem means that $B_0 = 1/3$.

In [14] (1962), M. Tomić uses an analogue of the argument in [12] (Sidon, 1927) to prove that

$$B_1 \geqslant \frac{1}{2}.$$

An elementary proof for this, which uses another result of L. Fejér ([16], 1914), can already be found in a note of E. Landau ([17], 1925). In [18] (1955), G. Ricci considered a wide class of problems of the same type and proved, among other results, the estimate

$$\frac{3}{5} < B_1 \leqslant \frac{1}{\sqrt{2}}.$$

The exact value

$$B_1 = \frac{1}{\sqrt{2}}$$

was given by E. Bombieri ([19], 1962). The estimate $B_1 \geqslant 3/5$ follows from Wiener's lemma, but the exact value $B_1 = 1/\sqrt{2}$ is obtained by another lemma of E. Landau, which is easily proved using Cauchy–Schwarz inequality:

Lemma 2 (E. Landau 1913)

$$\mathfrak{M}_0(r) \leqslant \frac{1}{\sqrt{1-r^2}},$$
$$\mathfrak{M}_1(r) \leqslant \frac{r}{\sqrt{1-r^2}}.$$

Note that the first estimate does not give any information about B_0, while the second provides the exact value of B_1.

1.3. More generalizations Estimates for the B_m's and the $B_{m,\alpha}$'s are given in ([18], 1955) and ([19], 1962) by G. Ricci and E. Bombieri; these results, including the above result about B_1, are ignored in most of recent papers about Bohr's theorem. Let us cite some results about $\mathfrak{M}_0(r)$:

(1) (Bombieri [19], 1962)

$$\mathfrak{M}_0(r) = \frac{3 - \sqrt{8(1-r^2)}}{r} \qquad \left(\frac{1}{3} \leqslant r \leqslant \frac{1}{\sqrt{2}}\right).$$

(2) (Bombieri–Bourgain [20], 2004)

$$\mathfrak{M}_0(r) < \frac{1}{\sqrt{1-r^2}} \qquad \left(\frac{1}{\sqrt{2}} < r < 1\right).$$

(3) (Bombieri–Bourgain [20], 2004) For $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, as $r \to 1-0$,

$$\mathfrak{M}_0(r) \geqslant \frac{1}{\sqrt{1-r^2}} - C(\varepsilon)\left(\ln\frac{1}{1-r}\right)^{(3/2)+\varepsilon}.$$

§ 2. H. Bohr's theorem for bounded symmetric domains

2.1. Let $\Omega \subset V$ be a bounded circled homogeneous domain in a finite dimensional complex vector space V. Denote by $\| \; \|_\Omega$ the spectral norm associated to Ω. For notations and results about complex bounded symmetric domains and their associated Jordan triple structure, see [21], [22].

The main result in [3] is Theorem 2, which is proved there for domains of type I (rectangular matrices) and for polydiscs.

This theorem is valid for any bounded circled homogeneous domain.

2.2. Differential of automorphisms. The proof in [3] depends on the following result, which is proved by the authors for classical domains of type I and IV, using *ad hoc* computations, but which is valid for any bounded circled homogeneous domain:

Theorem 3 *Let $\Omega \subset V$ be a bounded circled symmetric domain. Let $\|\cdot\|_\Omega$ be the associated spectral norm on V. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Then the operator norm of the derivative of ϕ at u is*

$$\|\mathrm{d}\,\phi(u)\|_\Omega = \frac{1}{1-\|u\|_\Omega^2}; \tag{4}$$

moreover,

$$\|\mathrm{d}\,\phi(u)\|_\Omega = \frac{\|u\|_\Omega}{1-\|u\|_\Omega^2}. \tag{5}$$

Sketch of proof. Denote by $\{\ ,\ ,\ \}$ the Jordan triple product on V, by $D(x,y)$ and $Q(x)$ the operators defined by $D(x,y)z = \{xyz\}$ and $2Q(x)y = \{xyx\}$, by

$$B(x,y) = \mathrm{id}_V - D(x,y) + Q(x)Q(y)$$

the *Bergman operator* (see [21]). For $u \in V$, denote by τ_u the translation $z \mapsto z+u$ and by $\widetilde{\tau}_u$ the rational map

$$\widetilde{\tau}_u(z) = z^u,$$

where z^u is the *quasi-inverse*

$$z^u = B(z,u)^{-1}\,(z - Q(z)u)\,,$$

defined in the open set of points z such that $B(z,u)$ is invertible.

For $u \in \Omega$, the map

$$\phi_u = \widetilde{\tau}_u \circ B(u,u)^{-1/2} \circ \tau_{-u} \tag{6}$$

is an automorphism of Ω which sends u to 0. The derivative of ϕ_u at u is then

$$\mathrm{d}\phi_u(u) = B(u,u)^{-1/2}. \tag{7}$$

(Recall that for $u \in \Omega$, the operator $B(u,u)$ is positive (with respect to the Hermitian scalar product on V: $(x \mid y) = \mathrm{tr}D(x,y)$), so that $B(u,u)^{-1/2}$ is well defined). Denote by $\| \ \|$ the Hermitian norm on V ($\|z\|^2 = \mathrm{tr}D(z,z)$) and by $\| \ \|$ the associated operator norm for linear endomorphisms of V. The *spectral norm* $\| \ \|_\Omega$ on V is given by

$$\|z\|_\Omega^2 = \|Q(z)\|\,; \tag{8}$$

the unit ball of this norm is Ω. We denote also by $\| \ \|_\Omega$ the associated operator norm for linear endomorphisms of V.

Using the *spectral decomposition* of u, the expression of $B(u,u)$ in the associated Peirce decomposition, the relation (8) and the relation

$$Q(B(u,u)^{-1/2}z) = B(u,u)^{-1/2}Q(z)B(u,u)^{-1/2}, \tag{9}$$

one obtains

$$\left\|B(u,u)^{-1/2}\right\|_\Omega = \left\|B(u,u)^{-1/2}\right\| = \frac{1}{1-\|u\|_\Omega^2}. \tag{10}$$

The theorem easily follows. □

Let Ω be a bounded circled symmetric domain. Let $f : \Omega \to \Omega$ be a holomorphic map. Denote by

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

its Taylor expansion in homogeneous polynomials f_k of order k. The following lemma generalizes Lemma 1 to bounded symmetric domains.

Lemma 3 *Let $u = f(0)$ and let $\phi \in \mathrm{Aut}\,\Omega$ such that $\phi(u) = 0$. Then*

$$\|\mathrm{d}\,\phi(u) \cdot f_k(z)\|_\Omega \leqslant \|z\|_\Omega^k\,. \tag{11}$$

See [5] for the proof. Along the same lines as in [3], one then proves, using this lemma:

Proposition 1 *Let Ω be a bounded circled symmetric domain. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Then*

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(u) \cdot f_k(z)\|_\Omega}{\|\mathrm{d}\,\phi(u)\|_\Omega} < 1 \tag{12}$$

for all $f : \Omega \to \Omega$ such that $f(0) = u$ and for all z such that $\|z\|_\Omega \leqslant (2 + \|u\|_\Omega)^{-1}$.

One proves also (see [5]):

Proposition 2 *Let Ω be a bounded circled symmetric domain. Let $u \in \Omega$ and let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(u) = 0$. Assume $\|u\|_\Omega > 1/3$ and $(1 + 2\,\|u\|_\Omega)^{-1} < a < 1$.*

Then there exist a holomorphic map $f : \Omega \to \Omega$ with $f(0) = u$ and $z \in \Omega$ with $\|z\|_\Omega = a$, such that

$$\sum_{k=0}^{\infty} \frac{\|\mathrm{d}\,\phi(u) \cdot f_k(z)\|_\Omega}{\|\mathrm{d}\,\phi(u)\|_\Omega} > 1.$$

Proof of Theorem 2. Proposition 1 shows that inequality (12) is satisfied for all maps $f : \Omega \to \Omega$ and all z such that $\|z\|_\Omega < 1/3$. As $(1 + 2\,\|u\|_\Omega)^{-1} \to 1/3$ as $\|u\|_\Omega \to 1 - 0$, proposition 2 implies that $1/3$ is the optimal bound. □

2.3. Some open questions.

Problem 2. Let $f : \Omega \to \Omega$ be a holomorphic map.

(1) With the assumption $f(0) = 0$, Theorem 2 gives

$$\sum_{k=0}^{\infty} \|f_k(z)\|_\Omega < 1$$

for all z such that $\|z\|_\Omega < 1/2$. Is the optimal bound equal to $1/\sqrt{2}$, as proved by E. Bombieri [19] in the one dimensional case?

(2) What is the same optimal bound for all maps f satisfying $f(0) = u$, with $u \in \Omega$ fixed? Propositions 1 and 2 show that this optimal bound belongs to the segment

$$\left[\frac{1}{2 + \|u\|_\Omega}, \frac{1}{1 + 2\,\|u\|_\Omega}\right].$$

In the one dimensional case, this is Ricci's estimate (see [18]). But Bombieri's results for the one dimensional case show that this estimate may be sharpened: in particular, the optimal bound in dimension one is $1/\sqrt{2}$ when $u = 0$, and $(1 + 2\,\|u\|_\Omega)^{-1}$ when $1/2 < \|u\|_\Omega < 1$.

For Ω a bounded circled symmetric domain, a natural problem generalizing H. Bohr's problem would be the following.

Problem 3. Let $\Omega \subset V$ be a bounded circled symmetric domain of rank r. Let $f : \Omega \to \Omega$ be a holomorphic map and consider the *Schmid decomposition* of its Taylor expansion

$$f(z) = \sum_{k_1 \geqslant \ldots \geqslant k_r \geqslant 0} f_{k_1 \cdots k_r}(z)$$

(where the $f_{k_1 \ldots k_r}$'s are polynomials in the irreducible K-modules for the linear subgroup K of $\operatorname{Aut}\Omega$). Determine the best constant C such that $z \in C\Omega$ ensures

$$\sum_{k_1 \geqslant \ldots k_r \geqslant 0} \|f_{k_1 \ldots k_r}(z)\|_\Omega < 1$$

for all holomorphic maps $f : \Omega \to \Omega$.

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