POISSON TRANSFORM FOR HYPERBOLOIDS

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§0. INTRODUCTION

The Poisson transform, as well as the Fourier transform and spherical functions, is one of the main tools of harmonic analysis on homogeneous spaces (in this connection, see [11]). For applications of it, see for example, [2], [3], [7], [8], [11], [12], [13], [15]. In this paper we consider this transform for hyperboloids $SO_0(p,q)/SO_0(p,q-1), q \geqslant 2$, in $\mathbb{R}^n$, an important class of semisimple symmetric spaces. Our main interest is an asymptotic behaviour of the Poisson transform at infinity. We write down explicit asymptotic decompositions of Poisson transform for arbitrary C^∞ functions. For K-finite functions these decompositions turn out to be absolutely convergent series. For "basic" functions (whose powers form the decompositions) we take both exponents and some other (perhaps more convenient) functions. In this paper we consider the general case $p > 1$, the case $p = 1$ (hyperboloids of one sheet) was studied in our paper [1].

§1. REPRESENTATIONS OF THE GROUP $SO_0(p,q)$ ASSOCIATED WITH A CONE

In this paragraph we recall some facts about representations of the group $G = SO_0(p,q)$ associated with a cone, see [9]. Besides it, we give explicit formulae for eigenvalues of an operator $A^{(r)}_{\sigma,\varepsilon}$, see (1.14) and (1.15). They generalize formulae obtained in [9] for eigenvalues of an intertwining operator $A_{\sigma,\varepsilon}$.

Let us consider in $\mathbb{R}^n$, $n = p + q$, the bilinear form

$$[x,y] = -x_1y_1 - \ldots - x_py_p + x_{p+1}y_{p+1} + \ldots + x_ny_n = \sum_{i=1}^n \lambda_i x_i y_i.$$

$\lambda_1 = \ldots = \lambda_p = -1, \lambda_{p+1} = \ldots = \lambda_n = 1, n = p + q, p \geqslant 2, q \geqslant 2$. The group G is the connected component of the identity of the group of all linear transformations of $\mathbb{R}^n$ preserving this form. We shall assume that G acts on $\mathbb{R}^n$ from the right: $x \mapsto xg$, hence we write vectors x in row form.

Consider two accompanying spaces $\mathbb{R}^p$ and $\mathbb{R}^q$ consisting of vectors $(x_1, \ldots, x_p)$ and $(x_{p+1}, \ldots, x_n)$. By $\langle\ ,\ \rangle$ we denote the standard inner product in both spaces. For $x \in \mathbb{R}^n$ we denote

$$|x| = \sqrt{x_1^2 + \ldots + x_p^2}.$$

Let S be the intersection of the cone $[x,x] = 0$ and the cylinder $|x| = 1$, $x \in \mathbb{R}^n$. It is the direct product $S_1 \times S_2$ of two unit spheres $S_1 = S^{p-1} \subset \mathbb{R}^p$ and $S_2 = S^{q-1} \subset \mathbb{R}^q$, so that a vector $s \in S$ is the pair

$$s = (u,v), \quad u \in \mathbb{R}^p, \quad v \in \mathbb{R}^q, \quad \langle u,u\rangle = 1, \quad \langle v,v\rangle = 1. \tag{1.1}$$

Let K denote the maximal compact subgroup of G preserving $|x| : K = SO(p) \times SO(q)$. It acts on S transitively, each component on its own sphere, so that S is a symmetric space of K, the stabilizer K_0 of the point $s^0 = (1,0,\ldots,0,1)$ is isomorphic to $SO(p-1) \times SO(q-1)$.

The Euclidean measure ds on S is the product $dudv$ of the Euclidean measures du, dv on the corresponding spheres. It is invariant with respect to K. The volume of S is equal to $\Omega_p\Omega_q$, where

$$\Omega_r = 2\pi^{r/2} \Big/ \Gamma(r/2).$$

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Let Δ_1, Δ_2 denote the Laplace-Beltrami operators on spheres S_1, S_2 respectively. Then the Laplace-Beltrami operator Δ_S on S is the sum of the operators Δ_1 and Δ_2.

Let R denote the representation of K by rotations in $\mathcal{D}(S) = C^\infty(S)$:

$$\Big(R(k)\varphi\Big)(s) = \varphi(sk), \ k \in K$$

It is unitary with respect to the inner product

$$(\psi, \varphi) = \int_S \psi(s)\overline{\varphi(s)} ds. \tag{1.2}$$

The representation R decomposes into a direct sum of pairwise non-equivalent irreducible representations ρ_z on spaces $H_z = H_l^{(p)} \otimes H_m^{(q)}$, where $z = (l, m)$ is a pair of integers called a *weight*. Here l ranges $\mathbb{Z}$ or $\mathbb{N} = \{0, 1, 2, ...\}$ for $p = 2$ or $p > 2$ respectively, similarly m ranges $\mathbb{Z}$ or $\mathbb{N}$. Therefore the lattice Z of weights z consists of integer points of the plane ($p = q = 2$), of the upper half-plane ($p = 2, q > 2$), of the right half-plane ($p > 2, q = 2$), of the first quadrant ($p > 2, q > 2$).

The space H_z is an eigenspace of the operators Δ_1 and Δ_2 and therefore of the operator Δ_S:

$$\Delta_1\varphi = \lambda_1\varphi, \ \ \Delta_2\varphi = \lambda_2\varphi, \ \ \Delta_S\varphi = \lambda_z\varphi, \ \ (\varphi \in H_z)$$

where

$$\lambda_1 = l(2 - p - l), \ \ \lambda_2 = m(2 - q - m), \ \ \lambda_z = \lambda_1 + \lambda_2.$$

For $p > 2, q > 2$ the spherical function in H_z with respect to K_0 is the product

$$\psi_z(s) = \psi_l^{(p)}(s_1)\ \psi_m^{(q)}(s_n), \tag{1.3}$$

where

$$\psi_m^{(r)}(t) = C_m^{\frac{r-2}{2}}(t) \Big/ C_m^{\frac{r-2}{2}}(1), \quad r > 2,$$

$C_m^\lambda(t)$ being the Gegenbauer polynomial. If $p = 2$ or $q = 2$ then the corresponding factor in (1.3) has to be replaced by $(s_1 + is_2)^l$ or $(s_{n-1} + is_n)^m$ respectively.

Let E_z be the projection operator in $\mathcal{D}(S)$ (or in $L^2(S)$) onto H_z:

$$\Big(E_z\varphi\Big)(s^0k) = \big(R(k)\varphi, \psi_z\big) \Big/ (\psi_z, \psi_z),$$

Let $\mathcal{D}_\varepsilon(S)$, $\varepsilon = 0, 1$, denote the subspace of functions $\varphi \in \mathcal{D}(S)$ of parity ε: $\varphi(-s) = (-1)^\varepsilon\varphi(s)$. Let $\sigma \in \mathbb{C}$. The representation $T_{\sigma,\varepsilon}$ of G acts on $\mathcal{D}_\varepsilon(S)$ in the following way:

$$\Big(T_{\sigma,\varepsilon}(g)\ \varphi\Big)(s) = \varphi\Big(\frac{sg}{|sg|}\Big)\ |sg|^\sigma.$$

It is continuous and indefinitely differentiable. The restriction of $T_{\sigma,\varepsilon}$ to K is the representation R_ε of K by rotations in $\mathcal{D}_\varepsilon(S)$. It is the direct sum of representations ρ_z with z from the lattice Z^ε : $l + m \equiv \varepsilon$. Here and further the sign $\equiv$ denotes the congruence modulo 2.

Denote

$$\sigma^* = 2 - n - \sigma.$$

The form (1.2) is invariant with respect to the pair $T_{\sigma,\varepsilon}$, $T_{\overline{\sigma}^*,\varepsilon}$, so that

$$\Big(T_{\sigma,\varepsilon}(g)\psi, \varphi\Big) = \Big(\psi, T_{\overline{\sigma}^*,\varepsilon}(g^{-1})\varphi\Big). \tag{1.4}$$

Let us denote by L_0 the following element of the Lie algebra $\mathfrak{g}$ of G:

$$L_0 = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Since L_0 centralizes K_0, the operator $T_{\sigma,\varepsilon}(L_0)$ preserves the family of spherical functions ψ_z. Namely, it carries ψ_z to a linear combination of four "neighboring" functions:

$$T_{\sigma,\varepsilon}(L_0)\psi_z = \sum_{i=1}^{4} \gamma_i(z)\ \beta_i(\sigma;z)\ \psi_{z+e_i}, \tag{1.5}$$

where $\gamma_i(z)$ are some positive functions, e_i are the following four vectors on the plane: $e_1 = (1,1)$, $e_2 = (1,-1)$, $e_3 = (-1,1)$, $e_4 = (-1,-1)$, and $\beta_i(\sigma;z)$ are the following functions of $z \in \mathbb{R}^2$:

$$\beta_1(\sigma;z) = \sigma - l - m, \quad \beta_2(\sigma;z) = \sigma - l + m + q - 2,$$

$$\beta_3(\sigma;z) = \sigma + l - m + p - 2, \quad \beta_4(\sigma;z) = \sigma + l + m + n - 4.$$

The line $\beta_i(\sigma;z) = 0$ on the plane z is called a barrier for $T_{\sigma,\varepsilon}$ if it meets Z^ε. If the line $\beta_i = 0$ is barrier, then $V_{\sigma,\varepsilon,i}$ denote the sum of subspaces H_z, $z \in Z^\varepsilon$, for which $\beta_i(\sigma;z) \geq 0$.

The subspaces $V_{\sigma,\varepsilon,i}$ are invariant with respect to G in the representation $T_{\sigma,\varepsilon}$. Any invariant subspace is the sum of intersections of spaces $V_{\sigma,\varepsilon,i}$. Therefore, if σ is not integer, then $T_{\sigma,\varepsilon}$ is irreducible.

Let $\Delta_{\mathfrak{g}}$ be the following element of the universal enveloping algebra of $\mathfrak{g}$ (it differs from the Casimir element by a factor only):

$$\Delta_{\mathfrak{g}} = \sum_{i<j} \lambda_i\lambda_j \Big(E_{ij} - \lambda_i\lambda_j E_{ji}\Big)^2,$$

E_{ij} being the standard matrix basis. The representation $T_{\sigma,\varepsilon}$ carries it to a scalar operator:

$$T_{\sigma,\varepsilon}(\Delta_{\mathfrak{g}}) = \sigma^*\sigma E. \tag{1.6}$$

We shall use the following notation for a character of $\mathbb{R}^*$:

$$x^{\lambda,\varepsilon} = |x|^\lambda \mathrm{sgn}^\varepsilon x = x_+^\lambda + (-1)^\varepsilon x_-^\lambda, \quad (\lambda \in \mathbb{C}, \varepsilon = 0,1).$$

Define an operator $A_{\sigma,\varepsilon}$ on $\mathcal{D}_\varepsilon(S)$:

$$\Big(A_{\sigma,\varepsilon}\varphi\Big)(s) = \int_S \big(-[s,\widetilde{s}]\big)^{\sigma^*,\varepsilon} \varphi(\widetilde{s})d\widetilde{s}. \tag{1.7}$$

The integral converges absolutely for $\mathrm{Re}\,\sigma < 3-n$ and can be extended to other σ by analyticity to a meromorphic function. The operator $A_{\sigma,\varepsilon}$ is a continuous operator in $\mathcal{D}_\varepsilon(S)$.

The operator $A_{\sigma,\varepsilon}$ intertwines the representations $T_{\sigma,\varepsilon}$ and $T_{\sigma^*,\varepsilon}$:

$$T_{\sigma^*,\varepsilon}A_{\sigma,\varepsilon} = A_{\sigma,\varepsilon}T_{\sigma,\varepsilon}.$$

On every subspace H_z it is a scalar operator:

$$A_{\sigma,\varepsilon}\ \varphi = a(\sigma,\varepsilon;z)\varphi, \quad \varphi \in H_z, \tag{1.8}$$

where

$$a(\sigma,\varepsilon;z) = 2^{\sigma+n}\pi^{\frac{n}{2}}(-1)^m \frac{\Gamma(3-n-\sigma)\Gamma\Big(\frac{2-n}{2}-\sigma\Big)}{\prod_{j=1}^4 \Gamma\Big(-\frac{1}{2}\beta_j(\sigma;z)\Big)}. \tag{1.9}$$

By (1.8) and (1.9) we have

$$A_{\sigma^*,\varepsilon}A_{\sigma,\varepsilon} = \gamma(\sigma,\varepsilon)E, \tag{1.10}$$

where

$$\gamma(\sigma,\varepsilon) = 2^n\pi^{n-4}\Gamma(\sigma+1)\Gamma(\sigma^*+1)\Gamma\Big(\frac{2-n}{2}-\sigma\Big)\Gamma\Big(\frac{2-n}{2}-\sigma^*\Big)\cdot$$

$$\cdot\Big[(-1)^\varepsilon\cos\frac{n\pi}{2} - \cos(\sigma+\frac{n}{2})\pi\Big]\cdot\Big[(-1)^{\varepsilon+p}\cos\frac{n\pi}{2} - \cos(\sigma+\frac{n}{2})\pi\Big].$$

Thus if σ is not integer, then the representations $T_{\sigma,\varepsilon}$ and $T_{\sigma^*,\varepsilon}$ are equivalent. In reducible case there is a partial equivalence.

The operator $A_{\sigma,\varepsilon}$ interacts with the form (1.2) as follows:

$$\left(A_{\sigma,\varepsilon}\psi,\varphi\right) = \left(\psi, A_{\overline{\sigma},\varepsilon}\varphi\right). \tag{1.11}$$

Let us extend the representation $T_{\sigma,\varepsilon}$ to the space $\mathcal{D}'_\varepsilon(S)$ of distributions on S of parity ε – by formula (1.4): now in (1.4) ψ is a distribution in $\mathcal{D}'_\varepsilon(S)$, φ is a function in $\mathcal{D}_\varepsilon(S)$, and (ψ,φ) means the value of a distribution ψ at a function φ. Indeed, it is an extension by attaching to a function $\psi \in \mathcal{D}_\varepsilon(S)$ the functional $\varphi \mapsto (\psi,\varphi)$ in $\mathcal{D}'_\varepsilon(S)$ by means of (1.2). Similarly we can extend the operator $A_{\sigma,\varepsilon}$ to $\mathcal{D}'_\varepsilon(S)$ – by (1.11).

In §5 we shall need the following operator $A^{(r)}_{\sigma,\varepsilon}$, $r \in \mathbb{N}$, in $\mathcal{D}_{\varepsilon+r}(S)$:

$$\left(A^{(r)}_{\sigma,\varepsilon}\varphi\right)(s) = \int_S \left(-[s,\widetilde{s}]\right)^{\sigma^*,\varepsilon} < u,\widetilde{u} >^r \ \varphi(\widetilde{s})d\widetilde{s}, \tag{1.12}$$

see (1.1), so that for $r=0$ it is (1.7). The operator $A^{(r)}_{\sigma,\varepsilon}$ intertwines the representation $R_{\varepsilon+r}$ of K with itself (but for $r>0$ it is not intertwining operator for G). Therefore the spaces H_z are eigenspaces of it:

$$A^{(r)}_{\sigma,\varepsilon}\ \varphi = a_r(\sigma,\varepsilon;z)\ \varphi, \quad \varphi \in H_z, \quad z \in Z^{\varepsilon+r}. \tag{1.13}$$

To write explicit expressions of a_r, we need to make some preparations. We shall use the "generalized powers":

$$a^{(m)} = a(a-1)...(a-m+1), \quad a^{[m]} = a(a+1)...(a+m-1).$$

Introduce the following polynomials $S_{rj}(l)$ in l of degree r:

$$S_{rj}(l) = \sum_k \frac{r^{(k)}}{2^{k-j}(2j-k)!(k-j)!} l^{(r-k)}$$

(the summation is taken over k satisfying inequalities $j \leqslant k \leqslant 2j,\ k \leqslant r$). Notice that these polynomials satisfy the following recurrence relation:

$$S_{rj} = S_{r-1,j-1} + (l+1+2j-r)S_{r-1,j}.$$

Theorem 1.1. *The eigenvalues $a_r(\sigma,\varepsilon;z)$ of the operator $A^{(r)}_{\sigma,\varepsilon}$ are given by the formulae:*

$$a_r(\sigma,\varepsilon;z) = 2^{\sigma+n}\pi^{\frac{n}{2}}(-1)^m \frac{\Gamma(3-n-\sigma)}{\Gamma\big(-\frac12\beta_3(\sigma-r;z)\big)\Gamma\big(-\frac12\beta_4(\sigma-r;z)\big)\Gamma\big(l+\frac{p}{2}\big)}\cdot$$

$$\cdot\left(\frac{d}{d\alpha}\right)^r\Big|_{\alpha=1}\alpha^l F\Big(1+\frac12\beta_3(\sigma-r;z), 1+\frac12\beta_4(\sigma-r;z); l+\frac{p}{2};\alpha^2\Big) \tag{1.14}$$

$$= 2^{\sigma+n-r}\pi^{\frac{n}{2}}(-1)^m \frac{\Gamma(3-n-\sigma)}{\Gamma\big(-\frac12\beta_1(\sigma-r;z)\big)\Gamma\big(-\frac12\beta_2(\sigma-r;z)\big)}\cdot$$

$$\cdot\sum_{j=0}^r 2^j S_{rj}(l)\frac{\Gamma\big(\frac{2-n}{2}-\sigma+r-j\big)}{\Gamma\big(-\frac12\beta_3(\sigma-r+2j;z)\big)\Gamma\big(-\frac12\beta_4(\sigma-r+2j;z)\big)} \tag{1.15}$$

(formula (1.14) is written for $p>2$, for $p=2$ one has to replace l by $|l|$).

Proof. To prove (1.14), we use the method from [9]. For definiteness, assume $p>2,\ q>2$. Putting in (1.13) $\varphi=\psi_z$ and taking it at $s=s^0$ (recall $\psi_z(s^0)=1$), we obtain by (1.12):

$$a_r(\sigma,\varepsilon;z) =$$

$$= \Omega_{p-1}\Omega_{q-1}\int_{-1}^1\int_{-1}^1 (s_1-s_n)^{\sigma^*,\varepsilon} s_1^r \psi_l^{(p)}(s_1)\psi_m^{(q)}(s_n)\cdot(1-s_1^2)^{\frac{p-3}{2}}(1-s_n^2)^{\frac{q-3}{2}}ds_1 ds_n.$$

Denote $\mu = (p-2)/2$, $\nu = (q-2)/2$ and compute separately the integrals

$$d_\pm = \Omega_{p-1}\Omega_{q-1}\int_{-1}^{1}\int_{-1}^{1}(x-y)_\pm^\lambda x^r \psi_l^{(p)}(x)(1-x^2)^{(p-3)/2}\psi_m^{(q)}(y)(1-y^2)^{(q-3)/2}dxdy.$$

Let us pass to Fourier transforms. The Fourier transform of the function x_+^λ is (see [5] p.196):

$$T(t) = i\Gamma(\lambda+1)\Big\{e^{i\lambda\pi/2}t_+^{-\lambda-1} - e^{-i\lambda\pi/2}t_-^{-\lambda-1}\Big\}. \tag{1.16}$$

Denote for brevity $\mu = (p-2)/2$, $\nu = (q-2)/2$. The Fourier transforms of the functions $(1-x^2)_+^{(p-3)/2}\psi_l^{(p)}(x)$ and $(1-y^2)_+^{(q-3)/2}\psi_m^{(q)}(y)$ are expressed by means of Bessel functions (see [6] 7.321) and are equal to

$$A(t) = i^l 2^\mu\sqrt{\pi}\Gamma\Big(\mu+1/2\Big)[t_+^{-\mu} + (-1)^l t_-^{-\mu}]J_{\mu+l}(|t|), \tag{1.17}$$

$$B(t) = i^m 2^\nu\sqrt{\pi}\Gamma\Big(\nu+1/2\Big)[t_+^{-\nu} + (-1)^m t_-^{-\nu}]J_{\nu+m}(|t|). \tag{1.18}$$

respectively. Therefore

$$d_+ = \frac{1}{2\pi}\Omega_{p-1}\Omega_{q-1}(-i)^r\int_{-\infty}^{\infty}T(t)A^{(r)}(-t)B(t)dt.$$

We can rewrite it as follows:

$$d_+ = \frac{1}{2\pi}\Omega_{p-1}\Omega_{q-1}i^r\left(\frac{d}{d\alpha}\right)^r\Big|_{\alpha=1}\int_{-\infty}^{\infty}t^{-r}T(t)A(-\alpha t)B(t)dt.$$

Introducing here (1.16), (1.17) and (1.18) we obtain

$$d_+ = (2\pi)^{\mu+\nu+1}i^{r+l+m+1}\Gamma(\lambda+1)\Big[e^{\frac{i\lambda\pi}{2}}(-1)^l - e^{-\frac{i\lambda\pi}{2}}(-1)^{m+r}\Big]\cdot$$

$$\cdot\left(\frac{d}{d\alpha}\right)^r\Big|_{\alpha=1}\alpha^{-\mu}\int_0^\infty t^{-\lambda-\mu-\nu-r-1}J_{\mu+l}(\alpha t)J_{\nu+m}(t)dt.$$

The last integral is computed with the help of formula [6] 6.574(1) and we finally obtain

$$d_+ = 2^{1-\lambda-r}\pi^{n/2}(-1)^m\frac{\Gamma(\lambda+1)}{\Gamma\big((\lambda-l+m+q+r)/2\big)\Gamma\big((\lambda-l-m+r+2)/2\big)\Gamma\big(l+p/2\big)}\cdot$$

$$\cdot\left(\frac{d}{d\alpha}\right)^r\Big|_{\alpha=1}\alpha^l F\Big(\frac{-\lambda+l+m-r}{2}, \frac{-\lambda+l-m-q+2-r}{2}; l+\frac{p}{2}; \alpha^2\Big).$$

Clearly $d_- = (-1)^{l+m+r}d_+$, so that for $z\in Z^{\varepsilon+r}$ we have $a_r(\sigma,\varepsilon;z) = d_+ + (-1)^\varepsilon d_- = 2d_+$ with $\lambda = \sigma^*$. It proves (1.14).

Now we make the differentiation in (1.14). We need some formulae from differential calculus (see, for example, [6] 0.433(1), 0.432(1), 0.431(1)):

$$\left(\frac{d}{xdx}\right)^r = \sum_{j=0}^{r-1} c_{rj}\ x^{-r-j}\ \left(\frac{d}{dx}\right)^{r-j},$$

$$\left(\frac{d}{dx}\right)^r = \sum_{j=0}^{[\frac{r}{2}]} d_{rj}\ x^{r-2j}\ \left(\frac{d}{xdx}\right)^{r-j}, \tag{1.19}$$

where

$$c_{rj} = (-1)^j\ \frac{(r-1+j)!}{2^j j!(r-1-j)!}; \tag{1.20}$$

$$d_{rj} = \frac{r^{(2j)}}{2^j j!} = \frac{r!}{2^j j!(r-2j)!}.$$

Applying (1.19) to the product $\alpha^l F(a,b;c;\alpha^2)$ where F is the hypergeometric function, we obtain

$$\left(\frac{d}{d\alpha}\right)^r\alpha^l F(a,b;c;\alpha^2)\Big|_{\alpha=1} =$$

$$= \frac{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}{\Gamma(c-a)\Gamma(c-b)}\sum_j 2^j\frac{\Gamma(c-a-b-j)}{\Gamma(1-a-j)\Gamma(1-b-j)}\sum_k\binom{r}{k}d_{k,k-j}l^{(l-k)}.$$

The last sum is precisely $S_{rj}(l)$. □

§2. Eigenfunctions of the Laplace-Beltrami operator

The hyperboloid $X : [x,x] = 1$ in $\mathbb{R}^n$ is a homogeneous space G/H, the stabilizer H of the point $x^0 = (0,...,0,1)$ is isomorphic $SO_0(p,q-1)$. Let us introduce on X the "polar" coordinates t, s $(t \in \mathbb{R}$, $s \in S)$ as follows: $x = (\mathrm{sh}t \cdot u, \mathrm{ch}t \cdot v)$, $s = (u,v)$, see (1.1). In these coordinates the operator Laplace-Beltrami Δ is:

$$\Delta = -\frac{\partial^2}{\partial t^2} - \big[(p-1)\mathrm{cth}t + (q-1)\mathrm{th}t\big]\frac{\partial}{\partial t} - \frac{\Delta_1}{\mathrm{sh}^2 t} + \frac{\Delta_2}{\mathrm{ch}^2 t}.$$

Notice that

$$\frac{x}{\mathrm{ch}t} \to (u,v) = s \quad (t \to +\infty), \tag{2.1}$$

so that S can be regarded as a boundary of X (in sense of Karpelevich).

Let U denote the representation of G on $C^\infty(X)$ by translations:

$$\Big(U(g)f\Big)(x) = f(xg).$$

It is continuous and indefinitely differentiable. It generates representations of Lie algebra of G and its universal enveloping algebra which we denote by the same symbol U. The element $\Delta_{\mathfrak{g}}$ is mapped just to Δ:

$$U(\Delta_{\mathfrak{g}}) = \Delta. \tag{2.2}$$

For $\sigma \in \mathbb{C}$, $\varepsilon = 0,1$, denote by $\mathcal{H}_{\sigma,\varepsilon}$ the subspace of $C^\infty(X)$ consisting of functions $f(x)$ satisfying

$$\Delta f = \sigma^* \sigma f, \quad f(-x) = (-1)^\varepsilon f(x). \tag{2.3}$$

It is closed. Clearly $\mathcal{H}_{\sigma,\varepsilon} = \mathcal{H}_{\sigma^*,\varepsilon}$. Let $U_{\sigma,\varepsilon}$ denote the restriction of U to $\mathcal{H}_{\sigma,\varepsilon}$.

Let us separate the variables t and s in (2.3): set $f(t,s) = R(t)\varphi(s)$. Then the function φ on S has to belong to a subspace H_z, $z = (l,m)$, see §1, and the function $R(t)$ on the real line must satisfy the equation

$$-\frac{d^2R}{dt^2} - \big[(p-1)\mathrm{cth}t + (q-1)\mathrm{th}t\big]\frac{dR}{dt} + \Big\{\frac{l(l+p-2)}{\mathrm{sh}^2 t} - \frac{m(m+q-2)}{\mathrm{ch}^2 t}\Big\}R = \sigma^*\sigma R. \tag{2.4}$$

Since $f(-t,s) = f(-t,u,v) = f(t,-u,v) = (-1)^l f(t,u,v) = (-1)^l f(t,s)$, the function R has parity l:

$$R(-t) = (-1)^l R(t). \tag{2.5}$$

and φ belongs to $\mathcal{D}_\varepsilon(S)$, so that $z \in Z^\varepsilon$.

For simplicity, we assume $p > 2$ (then $l \in \mathbb{N}$) in the main part of this section, and in the end of it we indicate the differences for $p = 2$.

Let us make the change of the variable and of the function in (2.4):

$$\mathrm{th}^2 t = y, \quad R = (\mathrm{ch}t)^\sigma (\mathrm{th}t)^l F,$$

then for F we obtain the hypergeometric equation with parameters $(-\sigma + l + m)/2$, $(-\sigma + l - m - q + 2)/2$, $l + p/2$. Notice that the first two parameters are precisely $-(1/2)\beta_1(\sigma;z)$ and $-(1/2)\beta_2(\sigma;z)$.

Thus the function

$$R(\sigma,z;t) = (\mathrm{ch}t)^\sigma (\mathrm{th}t)^l F\Big(\frac{-\sigma+l+m}{2}, \frac{-\sigma+l-m-q+2}{2}; l+\frac{p}{2}; \mathrm{th}^2 t\Big) = \tag{2.6}$$

$$= (\mathrm{ch}t)^\sigma\ (\mathrm{th}t)^l\ F\Big(-\frac{1}{2}\beta_1(\sigma,z), -\frac{1}{2}\beta_2(\sigma,z); l+\frac{p}{2}; \mathrm{th}^2 t\Big),$$

is a solution of (2.4) satisfying (2.5). It is invariant with respect to $m \mapsto 2 - q - m$ and $\sigma \mapsto \sigma^*$ (the last statement follows from [4] 2.1(23)).

Theorem 2.1. *For any function $f \in \mathcal{H}_{\sigma,\varepsilon}$ there exist functions φ_z from $H_z, z \in Z^\varepsilon$, such that*

$$f(t,s) = \sum_{z\in Z^\varepsilon} R(\sigma, z; t)\varphi_z(s). \tag{2.7}$$

This series converges with respect to the topology of the space $C^\infty(X)$. In particular, we can differentiate equation (2.7) with respect to t and apply the operators Δ_1 and Δ_2 as many times as desired.

The theorem is proved similarly to the corresponding theorem in [1].

Let us describe invariant subspaces in $\mathcal{H}_{\sigma,\varepsilon}$. Let $\mathcal{H}_{\sigma,\varepsilon,i}$ $(i = 1, 2, 3, 4)$ denote the subspace in $\mathcal{H}_{\sigma,\varepsilon}$ consisting of the functions f for which the functions φ_z from (2.7) are contained in $V_{\sigma,\varepsilon,i}$, see §1.

Theorem 2.2. *(see [15], [13]). The subspaces $\mathcal{H}_{\sigma,\varepsilon,i}$ and $\mathcal{H}_{\sigma^*,\varepsilon,i}$, $i = 1, 2$, are closed G-invariant subspaces in $\mathcal{H}_{\sigma,\varepsilon}$, and any such a subspace in $\mathcal{H}_{\sigma,\varepsilon}$ is a sum of intersecions of subspaces above. In particular, if σ is not integer, then $U_{\sigma,\varepsilon}$ is irreducible.*

For proof, it suffices to observe how the operator $U(L_0)$ acts on functions $h(\sigma, z; x) = R(\sigma, z; t)\psi_z(s)$, $z \in Z^\varepsilon$. For brevity, denote them h_z. Namely, as we shall see in §4,

$$U(L_0)h_z = \sum_{i=1}^{4} \lambda_i(\sigma, z)\, h_{z+e_i}, \tag{2.8}$$

where

$$\lambda_i(\sigma, z) = \begin{cases} \delta_i(z)\beta_i(\sigma, z)\beta_i(\sigma^*, z), & i = 1, 2, \\ \delta_i(z), & i = 3, 4, \end{cases} \tag{2.9}$$

and $\delta_i(z)$ are some non-zero numbers. □

Let us decompose $R(\sigma, z; t)$ in series in powers of $\eta = (\mathrm{ch}t)^{-2}$. Firstly we have:

$$R(\sigma, z; t) = g(\sigma, z)(\mathrm{ch}t)^\sigma W(\sigma, z; \eta) + g(\sigma^*, z)(\mathrm{ch}t)^{\sigma^*} W(\sigma^*, z; \eta), \tag{2.10}$$

where

$$W(\sigma, z; \eta) = (\mathrm{th}t)^l F\Big(\frac{-\sigma + l + m}{2}, \frac{-\sigma + l - m - q + 2}{2}; \frac{4-n}{2} - \sigma; \eta\Big), \tag{2.11}$$

$$g(\sigma, z) = \frac{\Gamma(\sigma + (n-2)/2)\Gamma(l + p/2)}{\Gamma((\sigma + l - m + p)/2)\Gamma((\sigma + l + m + n - 2)/2)}. \tag{2.12}$$

This formulae are obtained from (2.6) with the help of the transform [4] 2.10(1). Expanding for $t > 0$ the first factor $(\mathrm{th}t)^l = (1-\eta)^{l/2}$ in the right hand side of (2.11) in the binomial series, we can presented W as the sum of a series of powers of η:

$$W(\sigma, z; \eta) = \sum_{r=0}^{\infty} w_r(\sigma, z)\eta^r, \tag{2.13}$$

Further, expand W in a series of powers of $1 - \xi$ with $\xi = \mathrm{th}t$ and denote this sum by $V(\sigma, z; \xi)$:

$$V(\sigma, z; \xi) = \sum_{r=0}^{\infty} v_r(\sigma, z)(1-\xi)^r, \tag{2.14}$$

(Unfortunately, for $p > 1$ the function V is not expressed in terms of the hypergeometric function of $(1-\xi)/2)$. For $t > 0$ we have

$$V(\sigma, z; \xi) = W(\sigma, z; \eta), \quad \eta = 1 - \xi^2, \tag{2.15}$$

so that we can rewrite (2.10) for $t > 0$ in the following way

$$R(\sigma, z; t) = g(\sigma, z)(\mathrm{ch}t)^\sigma V(\sigma, z; \xi) + g(\sigma^*, z)(\mathrm{ch}t)^{\sigma^*} V(\sigma^*, z; \xi), \tag{2.16}$$

The coefficients v_r and w_r of series (2.13) and (2.14) are linked by formulae

$$w_r(\sigma, m) = \frac{1}{2^r r!} \sum_{j=0}^{r-1} (-1)^j c_{rj} (r-j)! v_{r-j}(\sigma, m), \tag{2.17}$$

$$v_r(\sigma, z) = \sum_{k=0}^{[r/2]} (-1)^k 2^{r-2k} \binom{r-k}{k} w_{r-k}(\sigma, z), \tag{2.18}$$

where c_{rj} are defined by (1.20).

Let us write v_r and w_r explicitly. Assume

$$\mathrm{Re}\sigma > r - (n-2)/2, \tag{2.19}$$

so that $\mathrm{Re}(\sigma^* - \sigma) < -2r$ and we have from (2.16):

$$\left(\frac{d}{d\xi}\right)^r V(\sigma, z; \xi)\Big|_{\xi=1} = \frac{1}{g(\sigma, z)} \left(\frac{d}{d\xi}\right)^r (\mathrm{ch}t)^{-\sigma} R(\sigma, z; t)\Big|_{\xi=1}.$$

Remembering (2.6), we obtain:

$$v_r(\sigma, z) = (-1)^r \frac{1}{r!} \frac{1}{g(\sigma, z)} \left(\frac{d}{d\xi}\right)^r \Big|_{\xi=1} \xi^l \cdot F\left(\frac{-\sigma + l + m}{2}, \frac{-\sigma + l - m - q + 2}{2}; l + \frac{p}{2}; \xi^2\right).$$

Comparing this with formulae (1.14) and (1.9), we obtain:

$$v_r(\sigma, z) = (-1)^r \binom{\sigma}{r} \frac{a_r(\sigma^* + r, \varepsilon + r, z)}{a(\sigma^*, z)}. \tag{2.20}$$

By analyticity in σ we can take off the restriction (2.19), so that (2.20) holds for all σ for which the right hand side of (2.20) makes sense.

Substituting into (2.20) explicit expressions (1.15), (1.9) we obtain

$$v_r(\sigma, z) = (-1)^r \frac{1}{r!} \sum_{j=0}^{r} 2^j S_{rj}(l) \frac{[(1/2)\beta_1(\sigma, z)]^{(j)} [(1/2)\beta_2(\sigma, z)]^{(j)}}{(\sigma + (n-4)/2)^{(j)}}. \tag{2.21}$$

Similarly

$$w_r(\sigma, z) = (-1)^r \frac{1}{r!} \sum_{j=0}^{r} \binom{r}{j} \left(\frac{l}{2}\right)^{(r-j)} \frac{[(1/2)\beta_1(\sigma, z)]^{(j)} [(1/2)\beta_2(\sigma, z)]^{(j)}}{(\sigma + (n-4)/2)^{(j)}}. \tag{2.22}$$

Notice that formulae (2.21) and (2.22) are connected by means of (2.17) and (2.18).

Lemma 2.3. *The coefficients $v_r(\sigma, z)$ and $w_r(\sigma, z)$ are polynomials in $\lambda_1 = l(2-p-l)$, $\lambda_2 = m(2-q-m)$ of degree r in λ_1 and λ_2 separately.*

The lemma follows from that (2.11) is invariant with respect to $m \mapsto 2 - q - m$ and $l \mapsto 2 - p - l$ (the latter statement follows from [4] 2.1(23)).

Let us normalize these polynomials so that the highest coefficient with respect to λ_2 (i.e. the coefficient of λ_2^r) is equal to 1. We denote these normalized polynomials by $\overset{0}{v}_r(\sigma; \lambda_1, \lambda_2)$ and $\overset{0}{w}_r(\sigma; \lambda_1, \lambda_2)$, so that

$$v_r(\sigma, z) = (-1)^r \frac{1}{2^r r! (\sigma + \frac{n-4}{2})^{(r)}} \overset{0}{v}_r(\sigma; \lambda_1, \lambda_2), \tag{2.23}$$

$$w_r(\sigma, z) = (-1)^r \frac{1}{2^{2r} r! (\sigma + \frac{n-4}{2})^{(r)}} \overset{0}{w}_r(\sigma; \lambda_1, \lambda_2) \tag{2.24}$$

Let us write some first polynomials $\overset{0}{v}_r$, $\overset{0}{w}_r$:

$$\overset{0}{v}_0 = \overset{0}{w}_0 = 1,$$

$$\overset{0}{v}_1 = \overset{0}{w}_1 = -\lambda_1 + \lambda_2 + \sigma(\sigma + q - 2),$$

$$\overset{0}{v}_2 = \big[-\lambda_1 + \lambda_2 + \sigma(\sigma + q - 2)\big]\big[-\lambda_1 + \lambda_2 + (\sigma - 1)(\sigma + q - 3) + p - 1\big] + 2(2\sigma + n - 4)\lambda_1,$$

$$\overset{0}{w}_2 = \big[-\lambda_1 + \lambda_2 + \sigma(\sigma + q - 2)\big]\big[-\lambda_1 + \lambda_2 + (\sigma - 2)(\sigma + q - 4)\big] + 2(2\sigma + n - 4)\lambda_1,$$

Let us consider the case $p = 2$. Then l ranges $\mathbb{Z}$ and we have to replace l by $|l|$ in formulae (2.6), (2.12), (2.14), (2.22). In formula (2.9) it is necessary to make an exception for $l = 0$, in which case we have

$$\lambda_i(\sigma, z) = \begin{cases} \delta_i(z)\ \beta_i(\sigma, z)\ \beta_i(\sigma^*, z), & i = 1, 2, \\ \delta_i(z),\ \beta_{i-2}(\sigma, z)\ \beta_{i-2}(\sigma^*, z), & i = 3, 4, \end{cases}$$

formulae (2.11), (2.21), (2.22) remain valid.

§3. H-INVARIANTS

We refer to [10]. Let us show elements invariant under H in representations of G described in §1. These invariants belong to $\mathcal{D}'(S)$ or to its subfactors.

Theorem 3.1. *The space of H-invariant elements from $\mathcal{D}_\varepsilon'(S)$ in the representation $T_{\sigma,\varepsilon}$ is one-dimensional except the case $q=2, \sigma=-m-1$, $m \in \mathbb{N}$, $m \equiv \varepsilon+1$, in which case this space is two-dimensional. For the generic case, a basis is the distribution*

$$\theta_{\sigma,\varepsilon}(s) = [x^0, s]^{\sigma,\varepsilon} = s_n^{\sigma,\varepsilon}.$$

or its first Laurent coefficient when it has a pole. In the exceptional case a basis consists, for example, of s_n^{-m-1} and $\delta^{(m)}(s_n)\mathrm{sgn}(s_{n-1})$.

The operator $A_{\sigma,\varepsilon}$ transfers $\theta_{\sigma,\varepsilon}$ to $\theta_{\sigma^*,\varepsilon}$ with a factor:

$$A_{\sigma,\varepsilon}\theta_{\sigma,\varepsilon} = j(\sigma,\varepsilon)\theta_{\sigma^*,\varepsilon}, \tag{3.1}$$

where

$$j(\sigma,\varepsilon) = 2^{1-\sigma}\pi^{\frac{n-4}{2}}\Gamma(\sigma+1)\Gamma\Big(\frac{2-n}{2}-\sigma\Big)\Big[(-1)^\varepsilon\cos\Big(\sigma+\frac{q}{2}\Big)\pi - \cos\frac{q\pi}{2}\Big]. \tag{3.2}$$

By (3.1) we have another expression for the factor γ from (1.10):

$$\gamma(\sigma,\varepsilon) = j(\sigma,\varepsilon)j(\sigma^*,\varepsilon). \tag{3.3}$$

§4. POISSON TRANSFORM

According to [11] the Poisson transform $P_{\sigma,\varepsilon}$ associated with the H-invariant $\theta_{\sigma,\varepsilon}$ is defined as follows

$$\Big(P_{\sigma,\varepsilon}\varphi\Big)(x) = \int_S \Big(T_\sigma(g^{-1})\theta_{\sigma,\varepsilon}\Big)(s)\varphi(s)ds = \int_S [x,s]^{\sigma,\varepsilon}\varphi(s)ds, \tag{4.1}$$

where $x = x^0 g$. It is a linear continuous operator from $\mathcal{D}_\varepsilon(S)$ to $C^\infty(X)$. (The continuity is proved similarly to [14] pp. 113–114). It intertwines $T_{\sigma^*,\varepsilon}$ with U:

$$U(g)\ P_{\sigma,\varepsilon} = P_{\sigma,\varepsilon}\ T_{\sigma^*,\varepsilon}(g), \quad g \in G. \tag{4.2}$$

Hence (see (1.6) and (2.2)):

$$\Delta \circ P_{\sigma,\varepsilon} = \sigma^*\sigma P_{\sigma,\varepsilon}.$$

The function $(P_{\sigma,\varepsilon}\varphi)(x)$ has parity ε:

$$\Big(P_{\sigma,\varepsilon}\varphi\Big)(-x) = (-1)^\varepsilon\Big(P_{\sigma,\varepsilon}\varphi\Big)(x),$$

so that the image of $P_{\sigma,\varepsilon}$ is contained in $\mathcal{H}_{\sigma,\varepsilon}$. The Poisson transform interacts with the operator $A_{\sigma,\varepsilon}$ as follows

$$P_{\sigma,\varepsilon}\ A_{\sigma,\varepsilon} = j(\sigma,\varepsilon)P_{\sigma^*,\varepsilon}. \tag{4.3}$$

As a function of σ, $P_{\sigma,\varepsilon}$ is a meromorphic function with poles at points $\sigma = -1-\varepsilon-2k, k \in \mathbb{N}$. So firstly we consider $\sigma \neq -1-\varepsilon-2k$.

Theorem 4.2. *Let $\varphi \in \mathcal{D}_\varepsilon(S)$. The decomposition (2.7) for the function $P_{\sigma,\varepsilon}\varphi \in \mathcal{H}_{\sigma,\varepsilon}$ have the form:*

$$\Big(P_{\sigma,\varepsilon}\varphi\Big)(t,s) = \sum_{z\in Z^\varepsilon} \chi(\sigma,z)\ R(\sigma,z;t)\ \Big(E_z\varphi\Big)(s), \tag{4.4}$$

where E_z are projection operators from §1, the numbers $\chi(\sigma,z)$ – let us call them the Poisson coefficients – are given for $p>2$ by the formula:

$$\chi(\sigma,z) = (-1)^l\ 2^{2-\sigma}\ \pi^{n/2}\ \frac{\Gamma(\sigma+1)}{\Gamma(l+p/2)\Gamma((\sigma-l-m+2)/2)\Gamma((\sigma-l+m+q)/2)}, \tag{4.5}$$

for $p=2$ one has to replace l by $|l|$. The radial factor from (4.4), i.e. the function

$$\Psi(\sigma,z;t)=\chi(\sigma,z)\ R(\sigma,z;t), \tag{4.6}$$

is expressed in the terms of W, V, see (2.11), (2.15), as follows

$$\Psi(\sigma,z;t)=(-1)^{\varepsilon}\Big\{(\mathrm{ch}t)^{\sigma}a(\sigma^*,z)W(\sigma,z;\eta)+(\mathrm{ch}t)^{\sigma^*}j(\sigma,\varepsilon)W(\sigma^*,z;\eta)\Big\}= \tag{4.7}$$

$$=(-1)^{\varepsilon}\Big\{(\mathrm{ch}t)^{\sigma}a(\sigma^*,z)V(\sigma,z;\xi)+(\mathrm{ch}t)^{\sigma^*}j(\sigma,\varepsilon)V(\sigma^*,z;\xi)\Big\}, \tag{4.8}$$

where a and j are given by formulae (1.9) and *(3.2) respectively.*

Proof. Expand the function $P_{\sigma,\varepsilon}\varphi\in\mathcal{H}_{\sigma,\varepsilon}$ into the series (2.7). Elements $k\in K$ preserve the coordinate t. Therefore, by (4.2) with $g=k$, we obtain that for every weight $z\in Z^{\varepsilon}$ the map $\varphi\mapsto\varphi_z$ of the space $\mathcal{D}_{\varepsilon}(S)$ into the space H_z commutes with rotations $k\in K$. Since the representations ρ_z (see §1) are irreducible and pairwise inequivalent, the map $\varphi\mapsto\varphi_z$ differs from E_z by the factor only – we denote its by $\chi(\sigma,z)$, – so that $\varphi_z=\chi(\sigma,z)\ E_z\varphi$, and we obtain the decomposition (4.4).

Now let us take $\varphi\in H_z,\ \ z\in Z^{\varepsilon}$. Then $\varphi=E_z\varphi$, and series (4.4) is reduced to one term only:

$$\Big(P_{\sigma,\varepsilon}\varphi\Big)(t,s)=\Psi(\sigma,z;t)\ \varphi(s), \tag{4.9}$$

where the function $\Psi(\sigma,z;t)$ is defined by (4.6). Introduce (2.10) in (4.6), we obtain:

$$\Psi(\sigma,z;t)=\chi(\sigma,z)\Big\{(\mathrm{ch}t)^{\sigma}g(\sigma,z)W(\sigma,z;\eta)+(\mathrm{ch}t)^{\sigma^*}g(\sigma^*,z)W(\sigma^*,z;\eta)\Big\}. \tag{4.10}$$

Assume that $\mathrm{Re}\sigma>(2-n)/2$. Then $\mathrm{Re}\sigma^*<(2-n)/2$, and by (4.10) and (2.11) we have

$$\Psi(\sigma,z;t)\sim(\mathrm{ch}t)^{\sigma}\chi(\sigma,z)g(\sigma,z)\quad(t\to+\infty). \tag{4.11}$$

On the other hand, the left hand side of (4.9) behaves as

$$(\mathrm{ch}t)^{\sigma}(-1)^{\varepsilon}\int_S\big(-[s,\widetilde{s}]\big)^{\sigma,\varepsilon}\varphi(\widetilde{s})d\widetilde{s}=(\mathrm{ch}t)^{\sigma}(-1)^{\varepsilon}a(\sigma^*;z)\varphi(s)$$

when $t\to+\infty$ (we used (2.1), (1.7), (1.8)), so that by (4.9) we have:

$$\Psi(\sigma,z;t)\sim(\mathrm{ch}t)^{\sigma}(-1)^{\varepsilon}a(\sigma^*;z)\quad(t\to+\infty). \tag{4.12}$$

Comparing (4.11) and (4.12) we obtain

$$(-1)^{\varepsilon}a(\sigma^*,z)=\chi(\sigma,z)g(\sigma,z). \tag{4.13}$$

Now by analyticity in σ we may take off the restriction $\mathrm{Re}\sigma>(2-n)/2$. Substituting expressions (1.9) of a and (2.12) of g into (4.13) we obtain (4.5).

Let us apply (4.3) to $\varphi\in H_z,\ z\in Z^{\varepsilon}$ and use (1.8) and (4.10). Comparing coefficients, we obtain

$$a(\sigma,z)\chi(\sigma,z)=j(\sigma,\varepsilon)\chi(\sigma^*,z). \tag{4.14}$$

From (4.13) and (4.14) we find that

$$\chi(\sigma,z)g(\sigma^*,z)=(-1)^{\varepsilon}j(\sigma,\varepsilon).$$

Together with (4.13) it proves (4.7) and (4.8). □

Formula (4.6) together with (4.5) can be proved by a direct computation. Integral representations obtained in this way will be used in §5. Namely, by (4.9) and (4.1) we have

$$\Psi(\sigma,z;t)=(\mathrm{ch}t)^{\sigma}\Omega_{p-1}\Omega_{q-1}\int_{-1}^{1}\int_{-1}^{1}(-\xi x+y)^{\sigma,\varepsilon}\psi_l(x)\psi_m(y)\ \cdot$$

$$\cdot(1-x^2)^{\frac{p-3}{2}}(1-y^2)^{\frac{q-3}{2}}dxdy, \tag{4.15}$$

where $\xi = \text{th}t$. As in §1, using Fourier transform, we obtain

$$\Psi(\sigma, z; t) = (\text{ch}t)^\sigma \Omega_{p-1}\Omega_{q-1}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[T(t) + (-1)^\varepsilon T(-t)\Big]A(\xi t)B(-t)dt,$$

where T, A, B are given by (1.16), (1.17), (1.18). So

$$\Psi(\sigma, z; t) = (\text{ch}t)^\sigma 2^{\mu+\nu+2}\pi^{\mu+\nu+1}i^{l+m+1}\Gamma(\sigma+1)\cdot$$

$$\cdot\left(e^{\frac{i\sigma\pi}{2}} - (-1)^\varepsilon e^{-\frac{i\sigma\pi}{2}}\right)\xi^{-\mu}\int_0^\infty t^{-\sigma-\mu-\nu-1}J_{\mu+l}(\xi t)J_{\nu+m}(t)dt. \tag{4.16}$$

The last integral is computed with the help of formula [6] 6.574(1), and we obtain (4.6) with χ given by (4.5).

Now let us prove (2.8). According to (4.2) the operator $U(L_0)$ acts on the functions $\Psi_z = P_{\sigma,\varepsilon}\psi_z$ in the same way as the operator $T_{\sigma^*,\varepsilon}(L_0)$ acts on the functions ψ_z (see (1.5)), i.e.

$$U(L_0)\Psi_z = \sum_{i=1}^{4}\gamma_i(z)\beta_i(\sigma^*, z)\Psi_{z+e_i}. \tag{4.17}$$

By (4.9) and (4.6) we have $\Psi_z = \chi(\sigma, z)h_z$. Substituting this into (4.17) we obtain

$$U(L_0)h_z = \sum_{i=1}^{4}\gamma_i(z)\beta_i(\sigma^*, z)\frac{\chi(\sigma, z+e_i)}{\chi(\sigma, z)}h_{z+e_i},$$

which is (2.8).

§5. Asymptotic behavior of the Poisson transform

First we expand the Poisson transform into series of powers of $1 - \text{th}t$ for the K-finite functions $\varphi \in \mathcal{D}(S)$, i.e. for linear combinations of functions from H_z. In this case the expansion is given by absolutely convergent series.

For $\sigma \in \mathbb{C}$, $r \in \mathbb{N}$, define the differential operators $L_{\sigma,r}$, $M_{\sigma,r}$ on S as follows. For $r > 0$ we set

$$L_{\sigma,r} = \overset{0}{v}_r(\sigma^*; \Delta_1, \Delta_2), \quad M_{\sigma,r} = \overset{0}{w}_r(\sigma^*; \Delta_1, \Delta_2),$$

where $\overset{0}{v}_r$, $\overset{0}{w}_r$ are the polynomials from §2, see (2.23), (2.24), Δ_1, Δ_2 are the Laplace-Beltrami operators on S_1, S_2, see §1; and for $r = 0$ we set $L_{\sigma,r} = 1$, $M_{\sigma,r} = 1$.

Theorem 5.1. *Let σ be generic: $\sigma \notin (n/2) + \mathbb{Z}$, $\sigma \notin -1-\varepsilon-2\mathbb{N}$. For any K-finite function $\varphi \in \mathcal{D}_\varepsilon(S)$ its Poisson transform $\left(P_{\sigma,\varepsilon}\varphi\right)(t,s)$ has the following expansion in series of powers of $1 - \text{th}t$:*

$$\left(P_{\sigma,\varepsilon}\varphi\right)(t,s) = (\text{ch}t)^\sigma\sum_{r=0}^{\infty}x_r(\sigma,\varepsilon)\left(A^{(r)}_{\sigma^*+r,\varepsilon-r}\varphi\right)(s)(1-\text{th}t)^r +$$

$$+(\text{ch}t)^{\sigma^*}\sum_{r=0}^{\infty}y_r(\sigma,\varepsilon)\left(L_{\sigma,r}\varphi\right)(s)(1-\text{th}t)^r, \tag{5.1}$$

where $A^{(r)}_{\lambda,\nu}$ is the operator from §1,

$$x_r(\sigma,\varepsilon) = (-1)^{\varepsilon+r}\binom{\sigma}{r},$$

$$y_r(\sigma,\varepsilon) = (-1)^\varepsilon j(\sigma,\varepsilon)\frac{1}{2^r r!(\sigma+\frac{n}{2})^{[r]}} =$$

$$= (-1)^r 2^{1-\sigma-r}\frac{1}{r!}\pi^{(n-4)/2}\Gamma(\sigma+1)\Gamma\Big(\frac{2-n}{2}-\sigma-r\Big)\Big[\cos\Big(\sigma+\frac{q}{2}\Big)\pi-\cos\Big(\varepsilon+\frac{q}{2}\Big)\pi\Big].$$

Both series in (5.1) converge absolutely.

Proof. It suffices to consider the case when $\varphi\in H_z,\ z\in Z^\varepsilon$. Then we have (4.9) and (4.8). Expand both functions V in (4.8) into series (2.14) of powers of $1-\xi$. Then

$$\Psi(\sigma,z;t)=$$

$$=(\mathrm{ch}t)^\sigma\sum_{r=0}^{\infty}(-1)^\varepsilon a(\sigma^*,z)v_r(\sigma,z)(1-\xi)^r+(\mathrm{ch}t)^{\sigma^*}\sum_{r=0}^{\infty}(-1)^\varepsilon j(\sigma,\varepsilon)v_r(\sigma^*,z)(1-\xi)^r.$$

Now applying (2.20) to the first series and (2.23) to the second series, we obtain

$$\Psi(\sigma,z;t)=(\mathrm{ch}t)^\sigma\sum_{r=0}^{\infty}x_r(\sigma,\varepsilon)a_r(\sigma^*+r,\varepsilon+r;z)(1-\xi)^r+$$

$$+(\mathrm{ch}t)^{\sigma^*}\sum_{r=0}^{\infty}y_r(\sigma,\varepsilon)\overset{0}{v}_r(\sigma^*;\lambda_1,\lambda_2)(1-\xi)^r,$$

whence (5.1) follows at once. The absolute convergence follows from the absolute convergence of the hypergeometric series (2.14) □

Equation (4.3) gives two relations for operators occurring in (5.1):

$$L_{\sigma,r}A_{\sigma,\varepsilon}=j(\sigma,\varepsilon)\frac{x_r(\sigma^*,\varepsilon)}{y_r(\sigma,\varepsilon)}A^{(r)}_{\sigma+r,\varepsilon+r},\tag{5.2}$$

$$A^{(r)}_{\sigma^*+r,\varepsilon+r}A_{\sigma,\varepsilon}=j(\sigma,\varepsilon)\frac{y_r(\sigma^*,\varepsilon)}{x_r(\sigma,\varepsilon)}L_{\sigma^*,r},\tag{5.3}$$

Denote by $\omega_r(\sigma)$ the coefficient in (5.2) (it does not depend on ε):

$$\omega_r(\sigma)=j(\sigma,\varepsilon)\frac{x_r(\sigma^*,\varepsilon)}{y_r(\sigma,\varepsilon)}=2^r(\sigma+n-2)^{[r]}\Big(\sigma+\frac{n}{2}\Big)^{[r]}\tag{5.4}$$

Notice that these two formulae (5.2), (5.3) are connected by means of (1.10), namely, multiplying (5.2) by $A_{\sigma^*,\varepsilon}$ from the right and using (1.10), (3.3) and replacing σ by σ^*, we obtain (5.3).

Let us take an *arbitrary* function $\varphi\in\mathcal{D}_\varepsilon(S)$ (not necessarily K-finite) and decompose it in a series of its Fourier components:

$$\varphi=\sum_{z\in Z^\varepsilon}E_z\varphi.$$

By the continuity of the Poisson transform we obtain

$$P_{\sigma,\varepsilon}\varphi=\sum_{z\in Z^\varepsilon}P_{\sigma,\varepsilon}(E_z\varphi).$$

Apply to each term here formulae (4.9) and (4.8), we represent $P_{\sigma,\varepsilon}\varphi$ as a sum:

$$\Big(P_{\sigma,\varepsilon}\varphi\Big)(t,s)=(\mathrm{ch}t)^\sigma\Big(P^+_{\sigma,\varepsilon}\varphi\Big)(\xi,s)+(\mathrm{ch}t)^{\sigma^*}\Big(P^-_{\sigma,\varepsilon}\varphi\Big)(\xi,s),$$

where $\xi=\mathrm{th}t$ and the operators $P^\pm_{\sigma,\varepsilon}$ are defined in the following way:

$$\Big(P^+_{\sigma,\varepsilon}\varphi\Big)(\xi,s)=\sum_{z\in Z^\varepsilon}(-1)^\varepsilon a(\sigma^*,z)V(\sigma,z;\xi)\Big(E_z\varphi\Big)(s),$$

$$\Big(P^-_{\sigma,\varepsilon}\varphi\Big)(\xi,s)=\sum_{z\in Z^\varepsilon}(-1)^\varepsilon j(\sigma,\varepsilon)V(\sigma^*,z;\xi)\Big(E_z\varphi\Big)(s).$$

For arbitrary function φ we can state only that the (5.1) has to be understood as an asymptotic expansion.

Theorem 5.2. *Let σ be generic: $\sigma \notin (n/2) + \mathbb{Z}$, $\sigma \notin -1-\varepsilon-\mathbb{N}$. For arbitrary function $\varphi \in \mathcal{D}_\varepsilon(S)$ its Poisson transform has the following asymptotic expansion for $t \to +\infty$:*

$$\left(P_{\sigma,\varepsilon}\varphi\right)(t,s) \sim (\mathrm{ch}t)^\sigma \sum_{r=0}^{\infty} x_r(\sigma,\varepsilon)\left(A^{(r)}_{\sigma^*+r,\varepsilon+r}\varphi\right)(s)(1-\mathrm{th}t)^r +$$

$$+(\mathrm{ch}t)^{\sigma^*} \sum_{r=0}^{\infty} y_r(\sigma,\varepsilon)\left(L_{\sigma,r}\varphi\right)(s)(1-\mathrm{th}t)^r. \tag{5.5}$$

The asymptotic equality (5.5) is understood as an asymptotic expansion of both functions $\left(P^{\pm}_{\sigma,\varepsilon}\varphi\right)(\xi,s)$ which means the following – for example, for $\left(P^{+}_{\sigma,\varepsilon}\varphi\right)$: for every $N \in \mathbb{N}$ there exists a constant C such that

$$\left|\left(P^{+}_{\sigma,\varepsilon}\varphi\right)(\xi,s) - \sum_{r=0}^{N} x_r(\sigma,\varepsilon)\left(A^{(r)}_{\sigma^*+r,\varepsilon+r}\varphi\right)(s)(1-\xi)^r\right| \leq C(1-\xi)^{N+1},$$

for all $\xi \in [0,\xi_0]$, $s \in S$, ξ_0 is some number from $(0,1)$.

The proof of Theorem 5.2 goes similarly to the case $p=1$, see [1]. Here we restrict ourselves to indicating some steps of the proof, cf [1].

Define the function $\Phi(\sigma,z;\xi)$ of ξ by the equation:

$$\Psi(\sigma,z;t) = (\mathrm{ch}t)^\sigma \Phi(\sigma,z;\xi),$$

where $z \in Z^\varepsilon$, $\xi = \mathrm{th}t$ and the function Ψ is given by (4.6) (or (4.7), (4.8)). Let us take in (4.9) $\varphi = \psi_z$ and $s = s^0$ then by definition (4.1) of the Poisson transform we obtain for Φ an integral representation, which is (4.15) where the factor $(\mathrm{ch}t)^\sigma$ has to be omitted. This integral converges absolutely for $\mathrm{Re}\sigma > -1/2$ and can be extended on the σ-plane as a meromorphic function. The function Φ has parity $\varepsilon + m \equiv l$. Denote

$$X(\sigma;z;\xi) = a(\sigma^*,z)V(\sigma,z;\xi),$$

where a and V are given by formulae (1.9) and (2.14) respectively.

Then we express X in terms of Φ:

$$X(\sigma;z;\xi) = (-1)^\varepsilon\left(\mu(\sigma,z)\Phi(\sigma,z;\xi) + \mu(\sigma,\widehat{z})\Phi(\sigma,\widehat{z};\xi)\right), \tag{5.6}$$

where $\widehat{z} = (\widehat{l},m)$, $\widehat{l} = 2-p-l$,

$$\mu(\sigma,z) \doteq \frac{\sin\left(\beta_3(\sigma,z)\pi/2\right)\cdot\sin\left(\beta_4(\sigma,z)\pi/2\right)}{\sin(\sigma+(n-2)/2)\pi\cdot\sin(l+p/2)\pi};$$

for even p one has to evaluate a undetermined form in (5.6). (Formula (5.6) is obtained from (2.11) by means of [4] 2.10(1)).

For $\mathrm{Re}\sigma > 0$ the function $\Phi(\sigma,z;\xi)$ in (5.6) is estimated by means of integral representation (4.15), the function $\Phi(\sigma,\widehat{z};\xi)$ in (5.6) is estimated by means of integral representation (4.16), where one has to replace l by $\widehat{l}$ and σ by σ^*. Thus, for $\mathrm{Re}\sigma > 0$ the function X is estimated by some constant independent of z.

Furhter, we have the following recursions for X:

$$X(\sigma;l,m;\xi) = X(\sigma;l,m-2;\xi) + \frac{2m+q-4}{\sigma+1}X(\sigma+1;l,m-1;\xi),$$

$$\frac{d}{d\xi}X(\sigma;l,m;\xi) = \frac{l}{2l+p-2}\sigma X(\sigma-1;l-1,m;\xi) + \sigma X(\sigma-1;l+1,m;\xi).$$

Applying these formulae, we can estimate X successively for all σ (first for $\mathrm{Re}\sigma > -1$, next for $-2 < \mathrm{Re}\sigma \leqslant -1$ etc.). This gives us desired estimates to prove (5.5). □

We can include $\sigma \in -1-\varepsilon-2\mathbb{N}$ (where $\theta_{\sigma,\varepsilon}$ has poles) dividing $\theta_{\sigma,\varepsilon}$ by a suitable function of σ, say, $\Gamma\Big((\sigma+1+\varepsilon)/2\Big)$. Then we have to divide by the same function both $P_{\sigma,\varepsilon}$ and $x_r(\sigma,\varepsilon)$, $y_r(\sigma,\varepsilon)$ in Theorems 5.1 and 5.2.

Similarly we consider the expansions of the Poisson transform in series (absolutely convergent for K-finite functions and asymptotic for arbitrary functions) of powers of $\eta=(\mathrm{ch}t)^{-2}$ and $\zeta=-e^{-2t}$. Then we replace operators $A^{(r)}_{\sigma^*+r,\varepsilon+r}$ and $L_{\sigma,r}$ by operators $2^{-r}B_{\sigma,\varepsilon,r}$ and $2^{-r}M_{\sigma,r}$ for η and operators $C_{\sigma,r}$ and $K_{\sigma,r}$ for ζ, where

$$B_{\sigma,\varepsilon,r}=\frac{1}{\omega_r(\sigma^*)}M_{\sigma^*,r}A_{\sigma^*,\varepsilon}=\sum_{j=0}^{r-1}\frac{c_{rj}}{(\sigma-r+1)^{[j]}}A^{(r-j)}_{\sigma^*+r-j,\varepsilon+r-j},$$

$$C_{\sigma,r}=\sum_{k=0}^{r}(-1)^{r-k}2^{-r+k}\binom{r}{k}A^{(k)}_{\sigma+k,\varepsilon+k},$$

$$K_{\sigma,r}=\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}(\sigma+k+n-2)^{[r-k]}\Big(\sigma+\frac{n}{2}\Big)L_{\sigma,k}$$

with ω_r, c_{rj} defined by formulae (5.4), (1.20). These pairs of operators are linked by relations similar to (5.2),(5.3), for example,

$$M_{\sigma,r}A_{\sigma,\varepsilon}=j(\sigma,\varepsilon)\frac{x_r(\sigma^*,\varepsilon)}{y_r(\sigma,\varepsilon)}B_{\sigma^*,\varepsilon,r},$$

$$B_{\sigma,\varepsilon,r}A_{\sigma,\varepsilon}=j(\sigma,\varepsilon)\frac{y_r(\sigma^*,\varepsilon)}{x_r(\sigma,\varepsilon)}M_{\sigma^*,r}.$$

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