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Functional-differential inclusions with impulses without switching convexity assumption [1]

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We consider functional-differential inclusions with multi-valued impulses. We do not suppose that the right hand side is convex with respect to switching. We introduce the notion of a generalized solution and prove that for a Cauchy problem with a Volterra operator (in sense of A. N. Tikhonov) a local generalized solution does exist and can be continued to a «maximal» interval. Finally we give a density principle for generalized solutions

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Functional-differential inclusions with impulses, see, for example, [1], [2], [3], have numerous and useful applications in mathematics and techniques. In the present work we consider inclusions whose right hand side is not supposed to be convex with respect to switching. We introduce notions of generalized solutions and generalized quasi-solutions of the Cauchy problem when the right hand side is a Volterra (in the sense of A. N. Tikhonov [4]) operator. We study the solvability and the extendability of generalized solutions and study connections between different families of solutions. Finally we give a density principle for generalized solutions.

Let $E \subset [a,b]$ be a Lebesgue measurable set, $\mathbf{L}^n(E)$ a space of summable (in Lebesgue sense) functions $x : E \to \mathbb{R}^n$ with the norm

$$\|x\|_{\mathbf{L}^n(E)} = \int_E |x(s)|ds,$$

$|\cdot|$ being the Euclidean norm in $\mathbb{R}^n$, and the corresponding distance $\rho_{\mathbf{L}^n(E)}$. Denote by $Q(\mathbf{L}^n[a,b])$ the family of nonempty closed sets of $\mathbf{L}^n[a,b]$ bounded by summable functions.

Fix m points $t_1, \ldots, t_m$ on $[a,b]$ satisfying $a < t_1 < \ldots < t_m < b$. Let us denote by $\widetilde{\mathbf{C}}^n[a,b]$ the space of functions $x(t)$ on $[a,b]$ with values in $\mathbb{R}^n$ that are continuous

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on each interval $[a, t_1]$, $(t_1, t_2]$, …, $(t_m, b]$ and have limits at points $t_1, \dots, t_m$ from the right. Equip this space with the norm

$$\|x\|_{\widetilde{\mathbf{C}}^n[a,b]} = \sup_{t \in [a,b]} |x(t)|.$$

For $\tau \in (a, b]$, let $\widetilde{\mathbf{C}}^n[a, \tau]$ be the space of restrictions to $[a, \tau]$ of functions in $\widetilde{\mathbf{C}}^n[a, b]$, the norm is given by the same formula with b replaced by τ.

Definition 1 A set $A \subset \mathbf{L}^n[a, b]$ is *convex with respect to switching*, if for any functions $x, y \in A$ and any measurable set $e \subset [a, b]$, the function $\chi_{(e)}\, x + \chi_{([a,b]\setminus e)}\, y$ belongs to A too. Here $\chi_{(g)}$ is the characteristic function of the set g. Let us denote by $\Pi(\mathbf{L}^n[a, b])$ the collection of sets in $\mathbf{L}^n[a, b]$ that are bounded, closed and convex with respect to switching, and by $\Omega\,[\Pi(\mathbf{L}^n[a, b])]$ its part consisting of convex sets.

For $A \subset \mathbf{L}^n[a, b]$, let swA (the "convex switching" hull of A) denote the set of functions $\chi_{(e)}\, x + \chi_{([a,b]\setminus e)}\, y$ with $x, y \in A$, and $\overline{sw}A$ the closure of swA in $\mathbf{L}^n[a, b]$.

Consider the problem

$$\dot{x} \in \Phi(x), \tag{1}$$

$$\Delta x(t_k) = I_k(x(t_k)),\ k = 1, \dots, m, \tag{2}$$

$$x(a) = x_0, \tag{3}$$

where a map $\Phi : \widetilde{\mathbf{C}}^n[a, b] \to Q(\mathbf{L}^n[a, b])$ meets the condition: for any bounded set $U \subset \widetilde{\mathbf{C}}^n[a, b]$ the image $\Phi(U)$ is bounded by a summable function, the maps $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous, $\Delta x(t_k) = x(t_k + 0) - x(t_k)$, $k = 1, 2, ..., m$.

Let us define the map $\widetilde{\Phi} : \widetilde{\mathbf{C}}^n[a, b] \to \Pi(\mathbf{L}^n[a, b])$ («convex with respect to switching») as

$$\widetilde{\Phi}(x) = \overline{sw}\Phi(x).$$

Definition 2 *A generalized solution* of problem (1)–(3) is a function $x \in \widetilde{\mathbf{C}}^n[a, b]$ for that there exists $q \in \widetilde{\Phi}(x)$ such that

$$x(t) = x_0 + \int_a^t q(s)\, ds + \sum_{k=1}^{m} \chi_{(t_k,b]}(t)\, \Delta x(t_k), \quad t \in [a, b]. \tag{4}$$

Let $\tau \in (a, b]$. For a set A of functions z on $[a, b]$, we denote by $A|_\tau$ the set of restrictions $z|_\tau$ functions z to $[a, \tau]$.

Definition 3 A map $\Phi : \widetilde{\mathbf{C}}^n[a, b] \to Q(\mathbf{L}^n[a, b])$ is a *Volterra operator in sense of Tikhonov* [4], if the condition $x|_\tau = y|_\tau$ implies $\Phi(x)|_\tau = \Phi(y)|_\tau$.

Let us define generalized solutions of (1)–(3) for intervals containing in $[a,b]$. First for $\tau \in (a,b]$ we define a continuous operator $V_\tau : \widetilde{\mathbf{C}}^n[a,\tau] \to \widetilde{\mathbf{C}}^n[a,b]$ by:

$$(V_\tau(x))(t) = \begin{cases} x(t), & \text{if} \quad t \in [a,\tau], \\ x(\tau), & \text{if} \quad t \in (\tau,b]. \end{cases}$$

For the *closed* interval $[a,\tau]$, a generalized solution of problem (1)–(3) is defined in the same way as for $[a,b]$ by Definition 2. Now $x \in \widetilde{\mathbf{C}}^n[a,\tau]$, the function q belongs to a set $\widetilde{\Phi}(V_\tau(x))|_\tau$ and the sum is taken over k for that $t_k \in [a,\tau]$.

Denote by $H(x_0,\tau)$ the set of all generalized solutions of (1)–(3) on $[a,\tau]$.

For the *half-open* interval $[a,c)$, a generalized solution of problem (1)–(3) is a function $x : [a,c) \to \mathbb{R}^n$ such that its restriction to any interval $[a,\tau]$, $a < \tau < c$, is a generalized solution of problem (1)–(3) on this interval $[a,\tau]$.

Definition 4 A generalized solution $x : [a,c) \to \mathbb{R}^n$ of problem (1)–(3) is called *noncontinuable*, if there exists no generalized solution y of problem (1)–(3) on the segment $[a,\tau]$, $t \in [c,b]$, such that $x(t) = y(t)$ for $t \in [a,c)$.

Theorem 1 *There exists $\tau \in (a,b]$ such that a solution of problem (1)–(3) exists on a segment $[a,\tau]$.*

Theorem 2 *The solution $x : [a,c) \to \mathbb{R}^n$ of (1)–(3) is continuable to a segment $[a,\tau]$, $\tau \in [c,b]$, if and only if $\overline{\lim}_{t\to c-0} |x(t)| < \infty$.*

Theorem 3 *Any generalized solution y of problem (1)–(3) on a segment $[a,\tau]$ can be continued to a noncontinuable solution x on $[a,c)$, $c \in (\tau,b]$, or on $[a,b]$.*

Definition 5 If there exists $r > 0$ such that for any $\tau \in (a,b]$ and any $y \in H(x_0,\tau)$ we have $\|y\|_{\widetilde{\mathbf{C}}^n[a,\tau]} \leqslant r$, then the set of all local generalized solutions of problem (1)–(3) is called *a priori bounded.*

Theorem 4 *Let the set of all local generalized solutions of problem (1)–(3) be a priori bounded. Then $H(x_0,\tau) \neq \varnothing$ for any $\tau \in (a,b]$.*

Definition 6 Let a function $\widehat{x} \in \widetilde{\mathbf{C}}^n[a,b]$ have representation (4) with x and q replaced by $\widehat{x}$ and $\widehat{q}$ respectively. Assume that this function $\widehat{x}$ is the limit in $\widetilde{\mathbf{C}}^n[a,b]$ of a sequence of functions x_i, $i = 1,2,\ldots$, having representation (4) with x and q replaced by x_i and q_i respectively such that $q_i \in \widetilde{\Phi}(\widehat{x})$ and $\Delta x_i(t_k)$, $k = 1,\ldots,m$, satisfy (2). Then the function $\widehat{x}$ is called a *generalized quasi-solution* of problem (1)–(3).

Let $\mathcal{H}(x_0)$ be the set of all generalized quasi-solutions of problem (1)–(3).

Define a map $\widetilde{\Phi}_{\rm co} : \widetilde{\mathbf{C}}^n[a,b] \to \Omega\,[\Pi(\mathbf{L}^n[a,b])]$ by $\widetilde{\Phi}_{\rm co}(x) = \overline{\rm co}\,\widetilde{\Phi}(x))$, the closure of the convex hull of $\widetilde{\Phi}(x)$.

Consider the problem

$$\dot{x} \in \widetilde{\Phi}_{\rm co}, \quad \Delta x(t_k) = I_k(x(t_k)), \quad x(a) = x_0. \tag{5}$$

Let $H_{\rm co}(x_0,\tau)$ denote the set of all solutions of (5) on $[a,\tau]$, $\tau \in (a,b]$.

Theorem 5 $\mathcal{H}(x_0) = H_{co}(x_0, b)$.

Definition 7 We say that impulses $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, and a map $\Phi : \widetilde{\mathbf{C}}^n[a, b] \to Q(\mathbf{L}^n[a, b])$ *possess the property* $\mathcal{A}$, if:

1) for each $k = 1, \ldots, m$ there exists a continuous non-decreasing function $\widetilde{I}_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\widetilde{I}_k(0) = 0$ and for any $x, y \in \mathbb{R}^n$ we have

$$|I_k(x) - I_k(y)| \leqslant \widetilde{I}_k(|x - y|);$$

2) there exists an isotonic continuous Volterra operator $\Gamma : \widetilde{\mathbf{C}}^1_+[a, b] \to \mathbf{L}^1_+[a, b]$, satisfying the following conditions:

(i) $\Gamma(0) = 0$,

(ii) for all $x, y \in \widetilde{\mathbf{C}}^n[a, b]$ and any measurable set $E \subset [a, b]$ we have

$$h_{\mathbf{L}^n(E)}[\Phi(x);\ \Phi(y)] \leqslant \|\Gamma(|x - y|)\|_{\mathbf{L}^1(E)}$$

(h denotes the Hausdorff distance),

(iii) the set of all solutions of

$$\dot{y} = \Gamma(y),\ \Delta y(t_k) = \widetilde{I}_k(y(t_k)),\ y(a) = 0$$

is a priori bounded.

Theorem 6 *Let the set of all locall generalized solutions of problem* (1) − (3) *be a priori bounded and the condition* $\mathcal{A}$ *is satisfied. Then* $H(x_0, b) \neq \varnothing$ *and the closure of* $H(x_0, b)$ *in* $\widetilde{\mathbf{C}}^n[a, b]$ *is precisely* $H_{\mathrm{co}}(x_0, b)$.

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