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Representations of clans and homogeneous cones

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We present a canonical way to construct an injective representation of a given clan (compact normal left symmetric algebra), which gives rise to the realization of a homogeneous cone in the space of real symmetric matrices by Rothaus [5] and Xu [8]

Keywords: homogeneous cones, clans, representations, real symmetric matrices

§ 1. Introduction

Since Vinberg [7] established the correspondence between homogeneous cones and clans (compact normal left symmetric algebra) with unit elements, the clans are convenient tools for the study of geometry and analysis on the homogeneous cones. The correspondence is in a sense similar to the one between Lie groups and Lie algebras, and also the one between symmetric cones and Euclidean Jordan algebras. In this article, we shall consider representations of clans and their relations to representations of homogeneous cones in the sense of Rothaus [5]. After some preliminaries in § 2, we see in § 3 that a representation of a clan is at the same time a representation of a homogeneous cone(Proposition 1), while every representation of a homogeneous cone is obtained as a composition of a representation of the associated clan with some linear transformations (Theorem 2). In § 4, we construct representations $R_1, \dots, R_r$ of a given clan V in a canonical way (Theorem 3). These R_k already appeared in [4, Section 4], and we refer some computational arguments about them to [4]. Taking the direct sum of $R_1, \dots, R_r$, we have an injective representation $\mathcal{R}$ of the clan V (Theorem 4). Then $\mathcal{R}$ gives a linear imbedding of the corresponding homogeneous cone into the space of real symmetric matrices, which coincides with the results by Rothaus [5] and Xu [8] eventually.

§ 2. Correspondence between clans and homogeneous cones

Let V be a finite dimensional real vector space, and $\Omega \subset V$ an open convex cone containing no line. We assume that Ω is homogeneous, that is, the group $\mathrm{GL}(\Omega) := \{g \in \mathrm{GL}(V)\,;\, g\Omega = \Omega\}$ acts on Ω transitively. By [7, Chapter 1, Theorem 1], there exists a connected split solvable Lie subgroup $H \subset \mathrm{GL}(\Omega)$ which acts on Ω

simply transitively. Such H is unique up to conjugacy in $\mathrm{GL}(\Omega)$. Let $\mathfrak{h} \subset \mathrm{End}(V)$ be the Lie algebra of H, and fix a point E in Ω. Then we have the linear isomorphism $\mathfrak{h} \ni L \mapsto L \cdot E \in V$ obtained by differentiating the orbit map $H \ni h \mapsto h \cdot E \in \Omega$. Thus, for an element $x \in V$, there exists a unique $L_x \in \mathfrak{h}$ for which $L_x \cdot E = x$. We define a bilinear multiplication $\triangle$ on V by $x \triangle y := L_x \cdot y \in V$, $x, y \in V$. Then the algebra $(V, \triangle)$ is a *clan (compact normal left symmetric algebra)* with a unit element E. Namely, the following axioms are satisfied:

(C1) Putting $[x \triangle y \triangle z] := x \triangle (y \triangle z) - (x \triangle y) \triangle z$ for $x, y, z \in V$, one has $[x \triangle y \triangle z] = [y \triangle x \triangle z]$ *(left symmetry).*

(C2) The bilinear form $(x|y) := \mathrm{tr}\, L_{x \triangle y}$ $(x, y \in V)$ defines a positive definite inner product on V *(compactness).*

(C3) For each $x \in V$, the linear operator L_x on V has only real eigenvalues *(normality).*

In terms of the clan structure $(V, \triangle)$, the cone Ω is described as $\Omega = \{(\exp L_x) \cdot E\,;\ x \in V\}$. Actually, if we start from any clan $(V, \triangle)$ with a unit element $E \in V$, we obtain a homogeneous cone in V this way, and the correspondence between the class of homogeneous cones and the class of clans with unit elements is one-to-one up to isomorphisms ([7, Chapter 2, Theorem 2]).

Let us give an example of a homogeneous cone. We denote by $\mathrm{Sym}(m, \mathbb{R})$ the space of real symmetric matrices of size m, and by $\mathcal{S}_m^+$ the subset of $\mathrm{Sym}(m, \mathbb{R})$ consisting of positive definite elements. Then $\mathcal{S}_m^+$ is a homogeneous cone, on which the group $\mathrm{GL}(m, \mathbb{R})$ acts transitively by $g \cdot x := g x\, {}^t g$, $x \in \mathcal{S}_m^+$, $g \in \mathrm{GL}(m, \mathbb{R})$). Let $H_m \subset \mathrm{GL}(m, \mathbb{R})$ be the group of lower triangular matrices with positive diagonals. Then the Lie algebra $\mathfrak{h}_m$ of H_m is the vector space of lower triangular matrices. For $x \in \mathrm{Sym}(m, \mathbb{R})$, we define $\underline{x} \in \mathfrak{h}_m$ by

$$(\underline{x})_{ij} := \begin{cases} 0 & (i < j), \\ x_{ii}/2 & (i = j), \\ x_{ij} & (i > j), \end{cases}$$

and set $\overline{x} := {}^t \underline{x}$. We choose the unit matrix I_m as the element E. Then the clan structure on $\mathrm{Sym}(m, \mathbb{R})$ associated to the cone $\mathcal{S}_m^+$ is given by

$$x \triangle y = \underline{x}\, y + y\, \overline{x}, \quad x, y \in \mathrm{Sym}(r, \mathbb{R}).$$

By a *representation of a clan* $(V, \triangle)$, we mean an algebra homomorphism from V to $\mathrm{Sym}(m, \mathbb{R})$ with some m, that is, a linear map $\phi : V \to \mathrm{Sym}(m, \mathbb{R})$ with the property

$$\phi(x \triangle y) = \underline{\phi(x)}\, \phi(y) + \phi(y)\, \overline{\phi(x)} \qquad (x,\, y \in V). \tag{2.1}$$

We require also $\phi(E) = I_m$ if $(V, \triangle)$ has a unit element E, which we always assume unless otherwise stated in what follows.

§ 3. Representations of homogeneous cones

We recall the notion of representation of a homogeneous cone introduced by Rothaus [5]. Let Ω be a homogeneous cone in a real vector space V. A linear map $\phi : V \to \mathrm{Sym}(m, \mathbb{R})$ is said to be a *representation of the homogeneous cone* Ω if the following two conditions are satisfied:

(R1) $\phi(\Omega) \subset \mathcal{S}_m^+$;

(R2) the group $G_\phi(\Omega)$ consisting of $g \in \mathrm{GL}(\Omega)$ for which there exist $\tilde{g} \in \mathrm{GL}(m, \mathbb{R})$, such that $\phi(g \cdot y) = \tilde{g}\, \phi(y)\, {}^t\tilde{g}$ for all $y \in V$, acts on Ω transitively.

Proposition 1 *Let $\Omega \subset V$ be a homogeneous cone, and $(V, \triangle)$ the associated clan. If a linear map $\phi : V \to \mathrm{Sym}(m, \mathbb{R})$ is a representation of the clan $(V, \triangle)$, then ϕ is a representation of the homogeneous cone Ω.*

Proof. We see from the relation (1) that

$$\phi((\exp L_x) \cdot y) = \left(\exp \underline{\phi(x)}\right) \phi(y) \left(\exp \overline{\phi(x)}\right), \quad x, y \in V. \tag{3.2}$$

Since $\mathfrak{h} = \{L_x; x \in V\}$ is split solvable, $\exp : \mathfrak{h} \to H$ is a diffeomorphism. Thus, for any $h \in H$, there exists $x \in V$ for which $h = \exp L_x$. Putting $\tilde{h} := \exp \underline{\phi(x)} \in H_m$, we have $\phi(h \cdot y) = \tilde{h}\, \phi(y)\, {}^t\tilde{h}$ $(y \in V)$, so that $h \in G_\phi$. Thus G_ϕ contains H, which acts on Ω transitively. On the other hand, we have $\phi(h \cdot E) = \tilde{h}\, {}^t\tilde{h} \in \mathcal{S}_m^+$. Therefore $\phi(\Omega) \subset \mathcal{S}_m^+$ and Proposition 1 is verified. □

Proposition 1 means that a representation of clan is automatically a representation of homogeneous cone. On the contrary, a representation of homogeneous cone is not necessarily a representation of clan in general. All representations of a homogeneous cone Ω are obtained from representations of the associated clans by taking compositions with linear automorphisms on the cones Ω and $\mathcal{S}_m^+$.

Theorem 2 *Let $\phi : V \to \mathrm{Sym}(m, \mathbb{R})$ be a representation of a homogeneous cone Ω. Then there exist elements $g_0 \in \mathrm{GL}(\Omega)$, $A_0 \in \mathrm{GL}(m, \mathbb{R})$ and a representation $\phi_0 : V \to \mathrm{Sym}(m, \mathbb{R})$ of the clan $(V, \triangle)$ for which*

$$\phi(x) = A_0\, \phi_0(g_0 \cdot x)\, {}^tA_0, \quad x \in V. \tag{3.3}$$

Proof. Let $G_\phi(\Omega, \mathbb{R}^m)$ be the group given by

$$G_\phi(\Omega, \mathbb{R}^m) := \left\{(g, \tilde{g}) \in \mathrm{GL}(\Omega) \times \mathrm{GL}(m, \mathbb{R}); \phi(g \cdot y) = \tilde{g}\, \phi(y)\, {}^t\tilde{g} \text{ for all } y \in V\right\}.$$

By definition, we have a surjective homomorphism $\pi : G_\phi(\Omega, \mathbb{R}^m) \to G_\phi(\Omega)$ mapping $(g, \tilde{g})$ to g. Let $\mathcal{G}$ be the identity component of $G_\phi(\Omega, \mathbb{R}^m)$. Since $\mathcal{G}$ is the identity component of an algebraic group, we have a decomposition $\mathcal{G} = \mathcal{H}\mathcal{K}$ with $\mathcal{H} \cap \mathcal{K} = \{e\}$, where $\mathcal{H}$ is a maximal connected split solvable subgroup and $\mathcal{K}$ is a maximal compact subgroup [6]. Without loss of generality, we can assume that $\mathrm{Ker}\, \pi \subset \mathcal{K}$. Then $\pi : \mathcal{H} \to \pi(\mathcal{H}) =: H_\phi \subset G_\phi$ is an isomorphism. In other words, we can define a representation $\sigma : H_\phi \to \mathrm{GL}(m, \mathbb{R})$ in such a way that $(h, \sigma(h)) \in \mathcal{H}$ for $h \in H_\phi$. Since H_ϕ is split solvable, the representation σ is simultaneously triangularizable, that is, there exists $A_1 \in \mathrm{GL}(m, \mathbb{R})$ for which $A_1\sigma(h)A_1^{-1} \in H_m$, $h \in H_\phi$. On the other hand, the groups H_ϕ and H are conjugate in $\mathrm{GL}(\Omega)$, so that we can take $g_1 \in \mathrm{GL}(\Omega)$ for which $H_\phi = g_1 H g_1^{-1}$. Noting that $A_1\, \phi(g_1 \cdot E)\, {}^tA_1 \in \mathcal{S}_m^+$, we take $A_2 \in H_m$ for which $A_1\, \phi(g_1 \cdot E)\, {}^tA_1 = A_2 {}^tA_2$. Setting $A_3 := A_2^{-1} A_1 \in \mathrm{GL}(m, \mathbb{R})$, we define $\sigma_0(h) := A_3\, \sigma(g_1 h g_1^{-1})\, A_3^{-1} \in \mathrm{GL}(m, \mathbb{R})$, $h \in H$, and $\phi_0(x) := A_3\, \phi(g_1 \cdot x)\, {}^tA_3 \in \mathrm{Sym}(m, \mathbb{R})$, $x \in V$. Then we have $\sigma_0(h) \in H_m$, $h \in H$, and $\phi_0(E) = I_m$. For $h \in H$ and $y \in V$ we observe

$$\begin{aligned}
\phi_0(h \cdot y) &= A_3\, \phi((g_1 h g_1^{-1}) \cdot (g_1 \cdot y))\, {}^tA_3 \\
&= A_3\, \sigma(g_1 h g_1^{-1})\, \phi(g_1 \cdot y)\, {}^t\sigma(g_1 h g_1^{-1})\, {}^tA_3 \\
&= \sigma_0(h)\, \phi_0(y)\, {}^t\sigma_0(h).
\end{aligned}$$

Differentiating this relation, we obtain

$$\phi_0(x \triangle y) = \dot{\sigma}_0(L_x)\phi_0(y) + \phi_0(y)^t\dot{\sigma}_0(L_x), \quad x, y \in V,$$

where $\dot{\sigma}_0$ is the differential representation of σ_0. In particular, since $\phi_0(E) = I_m$, we have

$$\phi_0(x) = \dot{\sigma}_0(L_x) + {}^t\dot{\sigma}_0(L_x).$$

Noting that $\dot{\sigma}_0(L_x) \in \mathfrak{h}_m$, we obtain $\dot{\sigma}_0(L_x) = \underline{\phi_0(x)}$. Therefore we have

$$\phi_0(x \triangle y) = \underline{\phi_0(x)}\, \phi_0(y) + \phi_0(y)\, \overline{\phi_0(x)},$$

which means that ϕ_0 is a representation of the clan $(V, \triangle)$. Putting $A_0 := A_3^{-1}$ and $g_0 := g_1^{-1}$, we have (3). □

§ 4. Canonical representation of a clan

Let $V = \sum^{\oplus}_{1 \leqslant j \leqslant k \leqslant r} V_{kj}$ be the normal decomposition of a clan V with respect to primitive idempotents $E_1, \ldots, E_r$ of V. Namely, we have $E = E_1 + \ldots + E_r$ and

$$V_{kj} = \left\{ x \in V; E_i \triangle x = \frac{1}{2}(\delta_{ik} + \delta_{ij})x,\ x \triangle E_i = \delta_{ij} x \text{ for } i = 1, \ldots, r \right\}. \tag{4.4}$$

Then we have $V_{kk} = \mathbb{R}E_k$. Moreover, the following multiplication relations hold:

$$\begin{gathered} V_{lk}\triangle V_{kj} \subset V_{lj}, \\ \text{if } k \neq i, j, \text{ then } V_{lk}\triangle V_{ij} = 0, \\ V_{lk}\triangle V_{mk} \subset V_{lm}, V_{ml} \text{ according to } l \geq m \text{ or } m \geq l. \end{gathered} \tag{4.5}$$

For $k = 1, \ldots, r$, we set $\mathcal{M}_k := V_{k1} \oplus \cdots \oplus V_{k,k-1} \oplus V_{kk}$. By (5) we have

$$\mathcal{M}_k\triangle\mathcal{M}_l \subset \mathcal{M}_l, \quad \mathcal{M}_l\triangle\mathcal{M}_k \subset \mathcal{M}_l$$

for $1 \leqslant k < l \leqslant r$. Therefore, if we set $\mathcal{I}_k := \mathcal{M}_k \oplus \ldots \oplus \mathcal{M}_r$, $k = 1, \ldots, r$, we have a two-sided ideal sequence

$$V = \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots \supset \mathcal{I}_r \supset \mathcal{I}_{r+1} := \{0\}.$$

The normal decomposition of V is orthogonal with respect to the inner product defined in the axiom (C2). Putting $n_{kj} := \dim V_{kj}$, $1 \leqslant j < k \leqslant r$, we take an orthonormal basis $\{\mathbf{f}_\alpha^{kj}\}$, $1 \leqslant \alpha \leqslant n_{kj}$, of the subspace V_{kj}. Put $m_k := \dim \mathcal{M}_k$, $k = 1, \ldots, r$. Noting that $m_k = n_{k1} + \ldots + n_{k,k-1} + 1$, we define a basis $\{\mathbf{e}_p^{(k)}\}$, $1 \leqslant p \leqslant m_k$, of $\mathcal{M}_k$ by

$$\mathbf{e}_p^{(k)} := \begin{cases} \mathbf{f}_\alpha^{kj}, & p = \alpha + \sum_{i<j} n_{ki}, \ 1 \leqslant \alpha \leqslant n_{kj}, \\ \sqrt{2}\|E_k\|^{-1}E_k, & p = m_k. \end{cases}$$

We remark that the basis $\{\mathbf{e}_p^{(k)}\}$ is not orthonormal because $\|\mathbf{e}\|_{m_k}^{(k)} = \sqrt{2}$. In view of the natural isomorphism $\mathcal{M}_k \ni v \mapsto \dot{v} := v + \mathcal{I}_{k+1} \in \mathcal{I}_k/\mathcal{I}_{k+1}$, we write $\dot{\mathcal{M}}_k$ for the quotient space $\mathcal{I}_k/\mathcal{I}_{k+1}$. For $x \in V$, let R_x be the right-multiplication operator on the clan V by x. We write $R_k(x)$ (resp. $L_k(x)$) for the matrix of the linear operator on $\dot{\mathcal{M}}_k$ induced by R_x (resp. L_x) with respect to the basis $\{\dot{\mathbf{e}}_p^{(k)}\}$, $1 \leqslant p \leqslant m_k$, of $\dot{\mathcal{M}}_k$. It is shown in [4, Lemma 4.2] that $R_k(x)$ is symmetric for $x \in V$.

Theorem 3 *The linear map $R_k : V \to \mathrm{Sym}(m_k, \mathbb{R})$ is a representation of the clan $(V, \triangle)$ for $k = 1, \ldots, r$.*

Proof. We define a linear form E_k^* on V by

$$\langle \sum_{i=1}^{r} x_i E_i + \sum_{i<j} x_{ji}\,, E_k^*\rangle := x_k, \quad x_i \in \mathbb{R}, \quad x_{ji} \in V_{ji}.$$

By [4, Lemma 4.2] we have

$$R_k(x) = L_k(x) + {}^tL_k(x) - \langle x\,,\, E_k^*\rangle\, I_{m_k}.$$

On the other hand, we see from (5) that $L_k(x)$ is a lower triangular matrix for $x \in V$. Thus we obtain

$$\underline{R_k(x)} = L_k(x) - \frac{1}{2}\,\langle x\,,\, E_k^*\rangle\, I_{m_k}.$$

On the other hand, we have by [4, (4.14)]

$$R_k(x \triangle y) = L_k(x)R_k(y) + R_k(y)^t L_k(x) - \langle x \, , \, E_k^* \rangle \, R_k(y).$$

Therefore we have

$$R_k(x \triangle y) = \underline{R_k(x)} R_k(y) + R_k(y)\overline{R_k(x)} \, , \quad x, \, y \in V.$$

Clearly $R_k(E) = I_{m_k}$, which completes the proof. □

Noting that $m_1 + \ldots + m_r = \dim V (=: n)$, we define a linear map $\mathcal{R} : V \to \mathrm{Sym}(n, \mathbb{R})$ by

$$\mathcal{R}(x) := \begin{pmatrix} R_1(x) & & & \\ & R_2(x) & & \\ & & \ddots & \\ & & & R_r(x) \end{pmatrix}, \quad x \in V.$$

Namely, $\mathcal{R}$ is the direct sum of the representations $R_1, \ldots, R_r$, so that $\mathcal{R}$ is also a representation of the clan $(V, \triangle)$.

Theorem 4 *The representation $\mathcal{R} : V \to \mathrm{Sym}(n, \mathbb{R})$ of the clan V is injective.*

Proof. Assume $\mathcal{R}(x) = 0$ with $x = \sum_{1 \leqslant i \leqslant j \leqslant r} x_{ji}$, $x_{ji} \in V_{ji}$. We see from [4, (4.11)] that $R_k(x) = 0$ yields $x_{ki} = 0$ for $i = 1, \ldots, k$. Therefore $x = 0$, whence Theorem 4 follows. □

Theorem 4 together with Proposition 1 implies that any homogeneous cone Ω is linearly imbedded into $\mathcal{S}_n^+$ with $n = \dim \Omega$. This description of the cone Ω is already given by [5] and [8] (see also [3] and [4]).

Let us consider the representations R_k, $k = 1, \ldots, r$, for the case that the clan V is $\mathrm{Sym}(r, \mathbb{R})$. Then the normal decomposition coincides with the natural decomposition by the entries. The inner product is given by $(x|y) = d \, \mathrm{tr} \, xy$ with $d = (r+1)/2$. The ideal $\mathcal{I}_k$, $k = 1, \ldots, r$, equals the set $\{x \in \mathrm{Sym}(r, \mathbb{R}); x_{ij} = 0 \text{ for all } i, j < k\}$. Noting that $\dim \mathcal{M}_k = k$, we take the basis of the module $\mathcal{M}_k$ as $\mathbf{e}_p^{(k)} := d^{-1/2}(E_{kp} + E_{pk})$, $1 \leqslant p < k$, and $\mathbf{e}_k^{(k)} := \sqrt{2} d^{-1/2} E_{kk}$, where E_{ij} is the (i, j)-matrix unit. Then we have

$$R_k(x) = (x_{ij})_{1 \leqslant i, j \leqslant k} \in \mathrm{Sym}(k, \mathbb{R}), \quad x = (x_{ij}) \in \mathrm{Sym}(r, \mathbb{R}).$$

It is easy to see directly that R_k is a representation of the clan $\mathrm{Sym}(r, \mathbb{R})$ in this case.

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