Hypergroup structures on C^*-algebras

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In the proposed talk, we suggest a generalization of the notion of a hypercomplex system with a compact basis [1] (compact hypergroup), give a theorem on existence of a Haar measure, a theorem of realization, and a Peter-Weyl type theorem. Examples of this generalization include compact quantum groups [3], quantum homogeneous spaces which arise when considering a pair of compact quantum groups H_1, H_2 and a Hopf $*$-algebra epimorphism $\pi : H_1 \to H_2$ [2].

Definition 1. Let $(A, \cdot, 1, *)$ be a C^* algebra with identity, $(A, \delta, \epsilon, \star)$ be a coinvolutive coalgebra with counit. Then $(A, \delta, \epsilon, \star)$ is called a *hypergroup structure* on the C^*-algebra A if δ is positive and

$$(a \cdot b)^\star = a^\star \cdot b^\star, \qquad \delta \circ * = (* \otimes *) \circ \delta,$$
$$\epsilon(a \cdot b) = \epsilon(a)\epsilon(b), \qquad \delta(1) = 1 \otimes 1,$$
$$\star \circ * = * \circ \star.$$

For a continuous functional on the C^*-algebra A, $\xi \in A^\circ$, define $\xi^+(a) = \overline{\xi(a^\star)}$, $a \in A$, and call an element $a \in A$ *positive definite* if $(\xi \otimes \xi^+) \circ \delta(a) \geq 0$ for all $\xi \in A^\circ$. A state ν is called a *Haar measure* if $(\nu \otimes id) \circ \delta(a) = (id \otimes \nu) \circ \delta(a) = \nu(a)1$.

Theorem 1. Let $(A, \delta, \epsilon, \star)$ be a hypergroups structure on the C^*-algebra A. Suppose that the linear space spanned by positive definite elements is dense in A. Then there exists a Haar measure ν, it is unique, and $\nu^+ = \nu$.

Definition 2. Let $(A, \delta, \epsilon, \star)$ be a hypergroup structure on a C^*-algebra $(A, \cdots, 1, *)$. We call $\mathcal{A} = (A, \cdot, 1, *, \delta, \epsilon, \star, \sigma_t)$ a *compact quantum hypergroup* if δ is completely positive, the linear span of positive definite elements is dense in A, and the following conditions hold:
(a) σ_t, $t \in \mathbf{R}$, is a continuous one-parameter group of automorphisms of A such that
(a_1) the set of analytic elements of A, $A_0 = \{a \in A : \|\sigma_z(a)\| < \infty \ \forall z \in \mathbf{C}\}$ is dense in A;
(a_2) A_0 is invariant with respect to $*$ and $\star$, and $\delta(A_0) \subset A_0 \otimes A_0$;
(a_3) the following relations hold on A_0 for all $z \in \mathbf{C}$:

$$\delta \circ \sigma_z = (\sigma_z \otimes \sigma_z) \circ \delta$$
$$\nu(\sigma_z(a)) = \nu(a);$$

(a_4) there exists $z_0 \in \mathbf{C}$ such that the Haar measure ν satisfies the following strong invariance condition for all $a, b \in A_0$:

$$(id \otimes \nu)\big[\big((* \circ \sigma_{z_0} \circ \star \otimes id) \circ \delta(a)\big) \cdot (1 \otimes b)\big] = (id \otimes \nu)\big((1 \otimes a) \cdot \delta(b)\big),$$

(b) the Haar measure ν is faithful on A_0.

Theorem 2. Let $\mathcal{A}$ be a compact quantum hypergroup and suppose that the C^*-algebra A is commutative. Let P denote the spectrum of A. Then P is the basis of a normal hypercomplex system $L_1(P, \nu)$ with a basis unit ϵ.

Define on A the L_2-norm by $\|a\|_\nu = \nu(a^* a)^{1/2}$.

Theorem 3. Let Q be the set of all finite dimensional nonequivalent left corepresentations of a compact quantum hypergroup $\mathcal{A}$ and $\mathcal{B} = \{t^q_{ij} : q \in Q,\ i, j = 1, \ldots, d_q = \dim V_q\}$ be the set of all matrix elements of these corepresentations with respect to some bases. Then the linear span of $\mathcal{B}$ is dense in A with respect to the norm $\|\cdot\|_\nu$.

References

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[3] S.L. Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111:613–665, 1987.