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Irreducible decompositions of unitary representations and extremal decompositions of positive definite functions on groups

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This paper concerns with positive definite functions (PDF) ϕ on the usual locally compact groups. A function ϕ has an extremal decomposition through an irreducible decomposition of the unitary representation corresponding to ϕ (by the Gelfand–Naimark–Segal construction method). However there are other ways to get extremal decompositions, for example via the Choqet theorem. So it is interesting to find conditions which distinguish the natural particular one from other decompositions. We describe a necessary and sufficient condition for the above problem as well as an interesting negative example related to [3], [4].

Keywords: locally compact groups, unitary representations, positive definite functions

§ 1. Introduction

The subject of this paper is a study of extremal decompositions of continuous positive definite functions (PDF) ϕ on a locally compact group G. A function ϕ has a unitary representation U of G corresponding to it through the Gelfand–Naimark–Segal construction method and after through an irreducible decomposition of U by the Mautner method, we have a natural extremal decomposition of ϕ. However, other disintegrations of ϕ are possible using, for example, the Choquet theorem. Hence, the question arises as to conditions under which a disintegration of ϕ coincides with the natural one described above. We describe a necessary and sufficient condition for the question as well as an interesting negative example related to [3], [4].

§ 2. Irreducible decompositions of unitary representations and extremal decompositions of PDF

2.1. Presentation of the problem. At the beginning of this section, we outline the problem that we will discuss in this paper. Let G be a separable locally compact group, and (H, U) be a continuous unitary representation of G with a normalized cyclic vector v. It is well-known that (H, U) can be decomposed irreducibly to $\{(H_\lambda, U_\lambda)\}_{\lambda\in\mathbb{R}}$ according to a factor decomposition of the ring $\mathcal{M}$ generated by $U(g)$, $g \in G$, and a maximal Abelian ring $\mathcal{A}$ of $\mathcal{M}'$ (cf. [1], p. 6):

$$U(g) \sim \sum U_\lambda(g) \quad \text{for all} \quad g \in G,$$

with a weight function $\sigma(\lambda)$ on $\mathbb{R}$ (cf. [2]). Then, we have

$$< U(g)v, v >_H = \int_{\mathbb{R}} < U_\lambda(g)v_\lambda, v_\lambda >_{H_\lambda} d\sigma(\lambda) \quad \text{with} \quad v = \int_{-} v_\lambda \sqrt{d\sigma(\lambda)}.$$

It is also well-known that $v_\lambda \neq 0$ for σ-a.e. λ, and this enables us to make the following definition. Denote

$$\rho(\lambda) = \|v_\lambda\|^2_{H_\lambda},$$

then

$$\phi(g) := < U(g)v, v >_H, \quad \phi_\lambda(g) := \rho(\lambda)^{-1} < U_\lambda(g)v_\lambda, v_\lambda >_{H_\lambda}, \quad d\mu(\lambda) := \rho(\lambda)d\sigma(\lambda).$$

Note that μ is a probability measure, and we have $\phi(e) = \phi_\lambda(e) = 1$, because v is normalized. Such a function is said to be normalized.

In any case, ϕ and ϕ_λ are continuous PDF on G, and further, ϕ_λ, $\lambda \in \mathbb{R}$, are extremal functions of a convex set of the normalized continuous PDFs by virtue of the irreducibility of (H_λ, U_λ). Hence, we arrive at an extemal decomposition of ϕ:

$$\phi(g) = \int_{\mathbb{R}} \phi_\lambda(g)d\mu(\lambda) \quad \text{for all} \quad g \in G. \tag{2.1}$$

Throughout this section, we refer to the disintegration (2.1) as a natural decomposition with cyclic vector v. Now, let us consider the converse problem.

Suppose that for a normalized continuous PDF ϕ on G, a disintegration:

$$\phi(g) = \int_X \phi_\lambda(g)d\mu(\lambda) \qquad \text{for all} \quad g \in G,$$

is given such that
- (P1) the measurable space $(X, \mathfrak{B})$ is standard (which is Borel isomorphic to the usual measurable space on $\mathbb{R}$), and μ is a probabilty measure on it (thus, on $\mathbb{R}$),
- (P2) for μ-a.e. λ, ϕ_λ is a normalized, continuous and extremal PDF on G,
- (P3) $\phi_\lambda(g)$ is measurable with respect to λ for each fixed $g \in G$.

We then ask
- (P) when and only when is this disintegration natural?

More precisely, let (H, U) be a continuous unitary representation of G with a normalized cyclic vector v corresponding to ϕ, and $\mathcal{M}$ be the ring generated by

$\{U(g)\}_{g\in G}$. Then we ask whether there exists a maximal Abelian ring $\mathcal{A}$ of $\mathcal{M}'$ with the following properties: take the ring $\mathcal{N}$ generated by $\mathcal{M}$ and $\mathcal{A}$, and using the center of $\mathcal{N}$, decompose H to a generalized direct sum of $H_\lambda, \lambda \in \mathbb{R}$, with a weight function $\sigma(\lambda)$; then, according to the irreducible decomposition of U and the decomposition of v:

$$U(g) \sim \sum U_\lambda(g), \quad \text{for all } \ g \in G, \quad \text{and} \quad v = \int_- v_\lambda \sqrt{d\sigma(\lambda)},$$

we have

$$\phi_\lambda(g) =< U_\lambda(g) v_\lambda, v_\lambda >_{H_\lambda} / \|v_\lambda\|^2_{H_\lambda} \quad \text{for all } \ g \in G \text{ and for } \ \sigma\text{-a.e. } \lambda,$$

and $d\mu(\lambda) = \|v_\lambda\|^2_{H_\lambda} d\sigma(\lambda)$.

The main issue that we wish to discuss in this section is that particular question.

2.2. Main results. In what follows, we consider the problem (P), and assume that all the conditions for (P) are fulfilled. In particular, in virtue of the above assumptions on the measurability, we may assume that $X = \mathbb{R}$, $\mathfrak{B} = \mathfrak{B}(\mathbb{R})$, and μ is a Borel probability measure on $\mathfrak{B}(\mathbb{R})$. Moreover, as mentioned above, we assume that G is separable, and thus, take a dense, countable subgroup $G_0 := \{g_1, g_2, \dots, g_n \dots\}$.

Now, take a continuous unitary representation (H, U) of G with a normalized cyclic vector $v \in H$ that correspondes to ϕ in (P) (for example, through the Gelfand–Naimark–Segal construction method). First, we consider a decomposition of H into a generalized direct sum (cf. [2], pp. 407–408). Set

$$N_m := \left\{ \xi = (\xi_1, \cdots, \xi_m) \in \mathbb{C}^m \;\middle|\; \sum_{i=1}^m \xi_i U(g_i) v = 0 \right\}.$$

Take a dense countable subset of N_m consisting of the vectors $\xi^{m,k} = (\xi_1^{m,k}, \dots, \xi_m^{m,k})$, $k \in \mathbb{N}$. It follows that

$$\int_{\mathbb{R}} \sum_{i,j=1}^m \xi_i^{m,k} \overline{\xi_j^{m,k}} \, \phi_\lambda(g_j^{-1} g_i) \, d\mu(\lambda) = \sum_{i,j=1}^m \xi_i^{m,k} \overline{\xi_j^{m,k}} < U(g_j^{-1} g_i) v, v >_H = 0.$$

Since the integrand in the above equality is non-negative, there exists a Borel set N with $\mu(N) = 0$ that satisfies for all $m \in \mathbb{N}$,

$$\sum_{i,j=1}^m \xi_i^{m,k} \overline{\xi_j^{m,k}} \, \phi_\lambda(g_j^{-1} g_i) = 0 \qquad \text{for all } \ \lambda \in N^c. \tag{2.2}$$

From this point on, we let λ run mainly through N^c. Take a countable space H_d consisting of linear combinations

$$\sum_{i=1}^n \alpha_i U(g_i) v,$$

where $n \in \mathbb{N}$, $\alpha_i = p_i + q_i\sqrt{-1}$, $p_i, q_i \in \mathbb{Q}$, $i = 1, \dots, n$. It is a dense subset of H. For $\lambda \in N$, introduce a scalar product $< \cdot, \cdot >_\lambda$ on H_d :

$$< \sum_{i=1}^{n} \alpha_i U(g_i)v, \sum_{i=1}^{m} \beta_i U(g_i)v >_\lambda := \sum_{i,j=1}^{n,m} \alpha_i \, \overline{\beta_j} \, \phi_\lambda(g_j^{-1} g_i).$$

To see that it is well-defined, we only have to check that

$$\sum_{i=1}^{n} \alpha_i U(g_i)v = 0 \quad \Longrightarrow \quad \sum_{i,j=1}^{n} \alpha_i \, \overline{\alpha_j} \, \phi_\lambda(g_j^{-1} g_i) = 0.$$

To this end, take a sequence ξ^{n,s_k}, $k \in \mathbb{N}$, that converges to $\alpha := (\alpha_1, \dots, \alpha_n)$, because α belongs to N_n. Since each component of ξ^{n,s_k} satisfies the equation (2.2), so do those of α.

Consequently, we get a Hilbert space H_λ, $\lambda \in N^c$, after completing the quotient space of H_d by the null kernel of the scalar product, and have a natural map from H_d to H_λ:

$$h := \sum_{i=1}^{n} \alpha_i U(g_i)v \longrightarrow h_\lambda.$$

It follows directly,

$$\|h_\lambda\|_{H_\lambda}^2 = \sum_{i,j=1}^{n} \alpha_i \, \overline{\alpha_j} \, \phi_\lambda(g_j^{-1} g_i).$$

Second, we go to the definition of $\sigma(\lambda)$-summability in [2].

An $\mathcal{F}$-family consists of the vector fields $\{f_\lambda\}_\lambda$ over $\mathbb{R}$ which fulfills the following conditions:

- (F1) for each $\lambda \in N^c$, $\quad f_\lambda \in H_\lambda$,
- (F2) $< f_\lambda, h_\lambda >_{H_\lambda}$ is a μ-measurable function of λ for each $h \in H_d$,
- (F3) $\|f_\lambda\|_{H_\lambda}$ is a μ-measurable function of λ and $\displaystyle\int \|f_\lambda\|_{H_\lambda}^2 d\mu(\lambda) < \infty$.

It is easy to see that a vector field $\{h_\lambda\}_\lambda$ derived from $h \in H_d$ by the natural maps (this will be denoted by $\{h_\lambda\}_\lambda$, $h \in H_d$, from this point on) belongs to $\mathcal{F}$. We also see that

$$\|h\|_H^2 = \int \|h_\lambda\|_{H_\lambda}^2 d\mu(\lambda).$$

Now, we ask whether $\mathcal{F}$-family with a suitable definition of the $\mathcal{F}$-integral, which is fully explained below, satisfies the conditions such that H is the generalized direct sum of H_λ with the weight function $\sigma(\lambda)$ (cf. [2]). First, we can easily see the following claim.

- (F4) $< f_\lambda, g_\lambda >_{H_\lambda}$ is μ-measurable for any $\{g_\lambda\}_\lambda \in \mathcal{F}$.

Next, let $\{f_\lambda\}_\lambda \in \mathcal{F}$, and put

$$L(h) := \int_{\mathbb{R}} < h_\lambda, f_\lambda >_{H_\lambda} d\mu(\lambda) \qquad \text{for any} \;\; \{h_\lambda\}_\lambda, \; h \in H_d.$$

Then $L(h)$ is a continuous linear functional on H_d that extends continuously to the entire space H. So we have a unique $f \in H$ such that

$$< h, f >_H = \int_{\mathbb{R}} < h_\lambda, f_\lambda >_{H_\lambda} d\mu(\lambda).$$

Note that $f = k$, if $\{f_\lambda\}_\lambda = \{k_\lambda\}_\lambda, k \in H_d$. We call f the $\mathcal{F}$-integral of $\{f_\lambda\}_\lambda$, and require that it be equal to the σ-integral in [2]. The following equality is crucial to this requirement:

• (F5) $\quad \|f\|_H^2 = \int_{\mathbb{R}} \|f_\lambda\|_{H_\lambda}^2 d\mu(\lambda) \quad$ for all $\{f_\lambda\}_\lambda \in \mathcal{F}$.

Once, we ensure that (F5) is fullfied, this enables us to define the isometric map T from the Hilbert space $\mathcal{F}$, equipped with the natural norm, to H. As the image of T contains a dense subset H_d, it is a surjection. Therefore, we find that with the definition of the $\mathcal{F}$-family and the $\mathcal{F}$-integral, H is the generalized direct sum of H_λ, if and only if the condition (F5) is fulfilled.

Theorem 2.1 *For* (F5) *to hold, it is necessary and sufficient that the following condition* (c.1) *is fulfilled.*

★ (c.1) *Suppose that* $\{f_\lambda\}_\lambda \in \mathcal{F}$ *satisfies*

$$\int_{\mathbb{R}} < h_\lambda, f_\lambda >_{H_\lambda} d\mu(\lambda) = 0 \qquad \text{for any } \{h_\lambda\}_\lambda,\ h \in H_d.$$

Then, we get $f_\lambda = 0$ *for* σ*-a.e.* λ.

We omit the proof because of space, but it is not difficult.

We remark that condition (c.1) has another expression. Namely, take a continuous unitary representation (K_λ, T_λ) of G with a cyclic vector t_λ such that

$$\phi_\lambda(g) =< T_\lambda(g)t_\lambda, t_\lambda >_{K_\lambda} \qquad \text{for all } g \in G.$$

As ϕ_λ is extremal and normalized, (K_λ, T_λ) is irreducible, and t_λ is a unit vector for each λ. Now, for the image h_λ of

$$h = \sum_{i=1}^{n} \alpha_i U(g_i) v \in H_d$$

by the natural map: $H_d \longrightarrow H_\lambda$, we get

$$\|h_\lambda\|_{H_\lambda}^2 = \sum_{i,j=1}^{n} \alpha_i\, \overline{\alpha_j}\, \phi_\lambda(g_j^{-1} g_i) = \left\| \sum_{i=1}^{n} \alpha_i\, T_\lambda(g_i)\, t_\lambda \right\|_{K_\lambda}^2 .$$

This enables us to define a unitary map from H_λ to $K_\lambda : h_\lambda \mapsto \sum_{i=1}^n \alpha_i T_\lambda(g_i) t_\lambda$. It follows that K_λ and the space of linear combinations $\sum_{i=1}^n \alpha_i T_\lambda(g_i) t_\lambda$, where $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $g_i \in G_0$, $i = 1, \ldots, n$, play a similar role to H_λ and $\{h_\lambda\}_\lambda, h \in H_d$, to obtain

$\sigma(\lambda)$-summability in [2]. Therefore, we now find that condition (c.1) is equivalent to the following condition (c.2).

★ (c.2) If a vector field $\{\eta_\lambda\}_\lambda, \eta \in K_\lambda$, over $\mathbb{R}$ satisfies the following three hypotheses:
(c.2.1) $< \eta_\lambda, T_\lambda(g)t_\lambda >_{K_\lambda}$ is μ-measurable for every $g \in G_0$,
(c.2.2) $\|\eta_\lambda\|_{K_\lambda}$ is μ-measurable, and $\int_{\mathbb{R}} \|\eta_\lambda\|^2_{K_\lambda} d\mu(\lambda) < \infty$, and
(c.2.3) $\int_{\mathbb{R}} < \eta_\lambda, T_\lambda(g)t_\lambda >_{K_\lambda} d\mu(\lambda) = 0$ for all $g \in G_0$,

then, $\eta_\lambda = 0$ for μ-a.e. λ.

Next we proceed to address the main problem in this subsection.

Theorem 2.2 *Let ϕ be a continuous, normalized PDF on a separable locally compact group G, and suppose that a disintegration of ϕ is given such that every condition in (P) is satisfied. Then the disintegration is natural, if and only if the condition* (c.2) *is fulfilled.*

Proof. Since the proof of necessity is not difficult, we go to the sufficiency. Given a disintegration in (P), take a continuous unitary representation (H, U) of G with a cyclic vector v, and take another irreducible ones (K_λ, U_λ) and normalized vectors $w_\lambda \in K_\lambda$ such that

$$\phi(g) =< U(g)v, v >_H, \text{ and } \phi_\lambda(g) =< U_\lambda(g)w_\lambda, w_\lambda >_{K_\lambda} \text{ for all } \lambda \in \mathbb{R} \text{ and } g \in G.$$

By virtue of condition (c.2) and Theorem 2.1 we get a representation of H as a generalized direct sum of K_λ, and every $U(g)v$ has an expression:

$$U(g)v = \int_{-} U_\lambda(g)w_\lambda \sqrt{d\mu(\lambda)},$$

first, for all $g \in G_0$, and second, for all $g \in G$. As $< U_\lambda(g)x_\lambda, y_\lambda >_{K\lambda}$ is measurable for any $x_\lambda = U_\lambda(\zeta_1)w_\lambda$, $y_\lambda = U_\lambda(\zeta_2)w_\lambda$, $\zeta_1, \zeta_2 \in G$, and $1 \geqslant \|U_\lambda(g)\|$, so $\sum U_\lambda(g)$ make sense (cf. [2]), and is equal to $U(g)$, because they coincides on H_d.

Therefore, the rest of the proof involves examining the ring $\mathcal{A}$ that is a center of the generalized direct sum. Recall that $\mathcal{M}$ is the ring generated by $U(g)$, $g \in G$. We only need to ensure that $\mathcal{A}$ is a maximal Abelian ring in $\mathcal{M}'$. Once this is assured, the ring $\mathcal{N}$ generated by $\mathcal{M}$ and $\mathcal{A}$ satisfies $\mathcal{N} \cap \mathcal{N}' = \mathcal{A}$, and it follows that the disintegration is natural.

Take any function $z \in \mathrm{L}^\infty_\mu(\mathbb{R})$, and define P_z on H by

$$P_z h := \int_{-} z(\lambda)h_\lambda \sqrt{d\mu(\lambda)} \quad \text{for all } h = \int_{-} h_\lambda \sqrt{d\mu(\lambda)}.$$

We know that $\mathcal{A} = \{P_z | \ z \in \mathrm{L}^\infty_\mu(\mathbb{R})\}$ (cf. [2]). Now, take any $P \in \mathcal{M}' \cap \mathcal{A}'$ which is a projection. Then, $< PP_B U(g)v, v >_H$ is an additive function of $B \in \mathfrak{B}$ for any

$g \in G$, where we use P_B instead of $P_{\chi B}$ for the sake of simplicity. We readily see that $P_B v = 0$ implies $< PP_B U(g)v, v >_H = 0$. In other words, the additive function is absolutely continuous with respect to μ. Hence, some $\omega_\lambda(g) \in \mathrm{L}^1_\mu(\mathbb{R})$ exists such that

$$< PP_B U(g)v, v >_H = \int_B \omega_\lambda(g) d\mu(\lambda) \qquad \text{for all} \;\; B \in \mathfrak{B}(\mathbb{R}). \tag{2.3}$$

Expanding both side of the inequality

$$\|P_B(I - U(g))v\|^2_H \geqslant \|PP_B(I - U(g))v\|^2_H,$$

and setting $Pv := \int_{-} p_\lambda \sqrt{d\mu(\lambda)}$ give

$$\int_B \{2 - \phi_\lambda(g) - \overline{\phi_\lambda(g)}\} d\mu(\lambda) \geqslant \int_B \{2\|p_\lambda\|^2_{K_\lambda} - \omega_\lambda(g) - \omega_\lambda(g^{-1})\} d\mu(\lambda) \geqslant 0. \tag{2.4}$$

It follows from (2.3) that

$$\int_B \omega_\lambda(g^{-1})\, d\mu(\lambda) = \int_B \overline{\omega_\lambda(g)}\, d\mu(\lambda),$$

and that $\overline{\omega_\lambda(g)} = \omega_\lambda(g^{-1})$ for any $g \in G$ and for μ-a.e. λ. Consequently, there exists a Borel set N_1 with $\mu(N_1) = 0$ such that

$$\overline{\omega_\lambda(g_k)} = \omega_\lambda(g_k^{-1}) \qquad \text{for all} \;\; k \in \mathbb{N} \;\; \text{and} \;\; \lambda \in N_1^c.$$

Hence, it follows from (2.4) that,

$$\int_B \{1 - \operatorname{Re} \phi_\lambda(g_k)\} d\mu(\lambda) \geqslant \int_B \{\|p_\lambda\|^2_{H_\lambda} - \operatorname{Re} \omega_\lambda(g_k)\} d\mu(\lambda) \;\; \text{for all} \;\; k \in \mathbb{N}.$$

Again, there exists a Borel set N_2 with $\mu(N_2) = 0$ such that

$$1 - \operatorname{Re} \phi_\lambda(g_k) \geqslant \|p_\lambda\|^2_{K_\lambda} - \operatorname{Re} \omega_\lambda(g_k) \text{ for all } \;\; k \in \mathbb{N} \;\; \text{and} \;\; \lambda \in N_2^c.$$

Proceeding in a similar manner, we find that there exists a Borel set N_3 with $\mu(N_3) = 0$ such that for any $\lambda \in N_3^c$, $\omega_\lambda(\cdot)$ is a PDF on G_0. Using $g := e$ in (2.4) produces

$$\|p_\lambda\|^2_{K_\lambda} = \omega_\lambda(e) \tag{2.5}$$

for μ-a.e. λ. Therefore, we conclude that there exists a Borel set N_4 with $\mu(N_4) = 0$ such that (2.5) and the following inequalities hold:

$$1 - \operatorname{Re} \phi_\lambda(g) \geqslant \omega_\lambda(e) - \operatorname{Re} \omega_\lambda(g) \quad \text{for all} \;\; g \in G_0, \tag{2.6}$$

$$\sum_{i,j=1}^n \alpha_i \overline{\alpha_j} \phi_\lambda(g_i^{-1} g_j) \geqslant \sum_{i,j=1}^n \alpha_i \overline{\alpha_j} \omega_\lambda(g_i^{-1} g_j) \geqslant 0, \tag{2.7}$$

for all $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $g_i \in G_0$, $i = 1, \ldots, n$.

In what follows, let λ run through only N_4^c, unless otherwise stated, and let us examine the continuity of $\omega_\lambda(\cdot)$. As $\phi_\lambda(g)$ is a continuous function of g, for each $n \in \mathbb{N}$, there exists a symmetric neighbourhood $U_{n,\lambda} \equiv U_n$ of e such that $U_{n+1}^2 \subset U_n$ and

$$n^{-1} \geqslant 1 - \operatorname{Re}\phi_\lambda(g) \qquad \text{for all} \quad g \in U_n.$$

Take any $g \in G$ and fix it. Further, take $\gamma_n \in G_0 \cap gU_n$ for each $n \in \mathbb{N}$. It follows that $\gamma_m^{-1}\gamma_n \subset U_N$ for all $n, m \geqslant N+1$, and that

$$\begin{aligned} \frac{2}{N}\omega_\lambda(e) &\geqslant 2\omega_\lambda(e)(1 - \operatorname{Re}\phi_\lambda(\gamma_m^{-1}\gamma_n)) \\ &\geqslant 2\omega_\lambda(e)(\omega_\lambda(e) - \operatorname{Re}\omega_\lambda(\gamma_m^{-1}\gamma_n)) \\ &\geqslant |\omega_\lambda(\gamma_m) - \omega_\lambda(\gamma_n)|. \end{aligned}$$

Hence, $\{\omega_\lambda(\gamma_n)\}_n$ converges, and the limit is independent of the choice $\{\gamma_n\}_n$. We denote the limit by the same letter $\omega_\lambda(g)$, because it is an extension of $\omega_\lambda(\cdot)$ from G_0 to the entire group.

Next, given $\epsilon > 0$, take $n \in \mathbb{N}$ such that

$$\frac{2}{n-1}\,\omega_\lambda(e) < \epsilon.$$

Then, for any $g' \in gU_n$ we can take the above $\{\gamma'_m\}_m$ from the set gU_n. Thus, we find that

$$\epsilon > \frac{2}{n-1}\,\omega_\lambda(e) \geqslant |\omega_\lambda(\gamma'_m) - \omega_\lambda(g)|.$$

Letting $m \longrightarrow \infty$, we obtain

$$\epsilon > \frac{2}{n-1}\,\omega_\lambda(e) \geqslant |\omega_\lambda(g') - \omega_\lambda(g)|,$$

and this demonstrates the continuity of $\omega_\lambda(\cdot)$.

Finally, for each $n \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^n \subset \mathbb{C}$, let F be the set consisting of $(g_1, \ldots, g_n)$, $g_i \in G$, such that

$$\sum_{i,j=1}^{n} \alpha_i\,\overline{\alpha_j}\,\phi_\lambda(g_i^{-1}g_j) \geqslant \sum_{i,j=1}^{n} \alpha_i\,\overline{\alpha_j}\,\omega_\lambda(g_i^{-1}g_j) \geqslant 0.$$

It is a closed set in G^n and contains G_0^n. Therefore, we have $F = G^n$, and both of the functions, ω_λ and $\phi_\lambda - \omega_\lambda$ are continuous PDFs on G. By the extremal assumption we thus have

$$\omega_\lambda(g) = \omega_\lambda(e)\phi_\lambda(g) \quad \text{for all} \quad g \in G \quad \text{and} \quad \lambda \in N_4^c.$$

It follows that regarding $\omega_\lambda(e) \equiv \omega(\lambda)$ as an essentially bounded measurable function of λ, we get

$$\begin{aligned} < P_\omega U(g)v, v >_H &= \int_{\mathbb{R}} \omega_\lambda(e) < U_\lambda(g)w_\lambda, w_\lambda >_{K_\lambda} \, d\mu(\lambda) \\ &= \int_{\mathbb{R}} \omega_\lambda(e)\phi_\lambda(g)\, d\mu(\lambda) \\ &= \int_{\mathbb{R}} \omega_\lambda(g)\, d\mu(\lambda) \\ &=< PU(g)v, v >_H \, . \end{aligned}$$

In other words, $P = P_\omega \in \mathcal{A}$. Therefore, we have $\mathcal{A} = \mathcal{M}' \cap \mathcal{A}'$, and $\mathcal{A}$ is a maximal ring of $\mathcal{M}'$. □

2.3. Characters on the infinite permutation group, and their disintegrations. We begin by reviewing briefly Thoma's result on characters. Let $\mathfrak{S}_\infty$ be the infinite permutation group of the finite permutations on $\mathbb{N}$. Note that each $g \in \mathfrak{S}_\infty$ has a cycle notation. In other word, g is expressed as a product of pairwise disjoint cycles. For each $n \geqslant 2$, let $r_n(g)$ be the number of the cycles with length n in the cycle notation for g.

By a character ϕ of $\mathfrak{S}_\infty$ we mean that it is a PDF on $\mathfrak{S}_\infty$ (equipped with the discrete topology) such that $\phi(e) = 1$ and $\phi(ghg^{-1}) = \phi(h)$ for any $g, h \in \mathfrak{S}_\infty$. Clearly, the set of the characters forms a convex set. The extremal point of the convex set is said to be indecomposable.

Finally, let $\ell_d^+(\mathbb{Z})$ be the set of the sequence $\{\beta_i\}_{i=-\infty}^{+\infty}$ such that $\beta_i \geqslant 0$ for all $i \in \mathbb{Z}$, two sequences $\{\beta_i\}_{i=1}^{+\infty}$ and $\{\beta_{-i}\}_{i=1}^{+\infty}$ are both decreasing for $i \in \mathbb{N}$, and $\sum_{i=-\infty}^{+\infty} \beta_i = 1$.

With that background, we are now ready to describe Thoma's result (cf. [4]).

Theorem 2.3 [4] *Given any indecomposable character ϕ on $\mathfrak{S}_\infty$, there exists a unique $\{\beta_i\}_{i=-\infty}^{+\infty} \in \ell_d^+(\mathbb{Z})$ such that*

$$\phi(g) = \prod_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} \beta_i^n + (-1)^{n+1} \sum_{i=1}^{\infty} \beta_{-i}^n \right)^{r_n(g)} \qquad \text{for all } \ g \in \mathfrak{S}_\infty.$$

Conversely, given $\{\beta_i\}_{i=-\infty}^{+\infty} \in \ell_d^+(\mathbb{Z})$, the right hand side of the above equality gives an indecomposable character.

Next, we recall again integral expressions of indecomposable characters on $\mathfrak{S}_\infty$ by Obata. However, we use a somewhat different notation from that of [3] to avoid inconsistencies with our previous notations.

Obata [3] had a disintegration of the indecomposable characters. Take the direct product $\mathbb{Z}^\infty$ of countable copies of $\mathbb{Z}$. An element $\lambda \in \mathbb{Z}^\infty$ is also regarded as a map from $\mathbb{N}$ to $\mathbb{Z}$. With each λ, we have a partition $\Gamma_\lambda := \{\lambda^{-1}(\{j\})\}_{j \in \mathbb{Z}}$ of $\mathbb{N}$. Define the Young subgroup H_λ by

$$H_\lambda := \prod_{j \in Z} \mathfrak{S}(\lambda^{-1}(\{j\})) \quad \text{(restricted direct product)},$$

where $\mathfrak{S}(\lambda^{-1}(\{j\}))$ is a group of the permutations that leaves every element in $\lambda^{-1}(\{j\})^c$ invariant. Take a one-dimensional representation χ_λ of H_λ defined by

$$\chi_\lambda(g) := \text{sgn } g_-,$$

according to the unique expression of $g = g_+ g_0 g_-$ with

$$g_+ \in \mathfrak{S}\left(\cup_{j\geqslant 1}\lambda^{-1}(\{j\})\right), \quad g_0 \in \mathfrak{S}(\lambda^{-1}(\{0\})), \quad g_- \in \mathfrak{S}\left(\cup_{j\geqslant 1}\lambda^{-1}(\{-j\})\right).$$

Take the induced representation $U(\Gamma_\lambda;\chi_\lambda) := \text{ind}(\chi_\lambda;\ H_\lambda \uparrow \mathfrak{S}_\infty)$. It is irreducible, if all the cardinals of the sets $\lambda^{-1}(\{n\})$, $n \in \mathbb{N}$, are infinite. In any case, it is a cyclic unitary representation for any $\lambda \in \mathbb{Z}^\infty$, and

$$\phi_\lambda(g) := \begin{cases} \chi_\lambda(g), & \text{if } g \in H_\lambda, \\ 0, & \text{otherwise} \end{cases}$$

is a normalized PDF that corresponds to $U(\Gamma_\lambda;\chi_\lambda)$ with a cyclic vector. Finally, we introduce a probability measure $\mu_\beta \equiv \mu$ for each $\beta := \{\beta_i\}_{i=-\infty}^{+\infty} \in \ell_d^+(\mathbb{Z})$ on the standard Borel space $(\mathbb{Z}^\infty, \mathfrak{B}(\mathbb{Z}^\infty))$ as a product measure of countable copies of ν on $\mathbb{Z}$ such that $\nu(\{i\}) = \beta_i$ for all $i \in \mathbb{Z}$. Note that the set of $\lambda \in \mathbb{Z}^\infty$ such that $|\lambda^{-1}(\{n\}| < \infty$ is of zero measure, so that $U(\Gamma_\lambda;\chi_\lambda)$ is irreducible for μ-a.e. λ. In other word, ϕ_λ is extremal for the same λ.

Theorem 2.4 [3] *Let $\phi \equiv \phi_\beta$ be an indecomposable character corresponding to $\beta \in \ell_d^+(\mathbb{Z})$ with $\beta_0 = 0$. Then, we have*

$$\phi_\beta(g) = \int_{\mathbb{Z}^\infty} \phi_\lambda(g)\, d\mu_\beta(\lambda) \quad \textit{for all} \quad g \in \mathfrak{S}_\infty.$$

In what follows, we examine whether Obata's disintegration is natural or not. We retain the notation in the previous subsections.

So, given $\beta \in \ell_d^+(\mathbb{Z})$, $\beta_0 = 0$, we have the indecomposable character $\phi \equiv \phi_\beta$, and the probability measure $\mu \equiv \mu_\beta$ on $\mathbb{Z}^\infty$, and the disintegration described in Theorem 2.4. The representation space $\mathcal{H}'_\lambda$ of $U(\Gamma_\lambda;\chi_\lambda)$ consists of the $\mathbb{C}$-valued functions on $\mathfrak{S}_\infty$ which satisfies

(1) $f(gh) = \chi_\lambda(h) f(g)$ for all $g \in \mathfrak{S}$ and $h \in H_\lambda$,

(2) $\sum_{\overline{g}\in G/H_\lambda} |f(\overline{g})|^2 < \infty$.

As usual, we regard $|f(\overline{g})|$ as a function on $\mathfrak{S}/H_\lambda$ in virtue of (1). The representation $U(\Gamma_\lambda;\chi_\lambda)$ acts by translations:

$$U(\Gamma_\lambda;\chi_\lambda)(g)f(\cdot) = f(g^{-1}\cdot) \quad \text{for all} \quad f \in \mathcal{H}'_\lambda, \quad g \in \mathfrak{S}_\infty.$$

Now, we deform $(\mathcal{H}'_\lambda, U(\Gamma_\lambda;\chi_\lambda))$, using a section $s \equiv s_\lambda$ of the natural map π; $\mathfrak{S}_\infty \longrightarrow \mathfrak{S}_\infty/H_\lambda$. For any $f \in \mathcal{H}'_\lambda$ put

$$F(X) := f(s(X)) \qquad \text{for all} \quad X \in \mathfrak{S}_\infty/H_\lambda.$$

We readily see that $F \in \ell^2(\mathfrak{S}_\infty/H_\lambda)$, and the map $W : f \longrightarrow F$ is isometric. Moreover, for any $F \in \ell^2(\mathfrak{S}_\infty/H_\lambda)$, a function f defined by

$$f(g) := F(\pi(g))\chi_\lambda(s(\pi(g))^{-1}g)$$

satisfies $f(s(X)) = F(X)$ for any $X \in \mathfrak{S}_\infty/H_\lambda$ and $f(gh) = \chi_\lambda(h)f(g)$ for all $g \in \mathfrak{S}_\infty$ and $h \in H_\lambda$. In other words, W is a unitary operator. We set

$$U_\lambda(g) := W \circ U(\Gamma_\lambda; \chi_\lambda)(g) \circ W^{-1} \quad \text{and} \quad K_\lambda := \ell^2(\mathfrak{S}_\infty/H_\lambda).$$

Then, it easily follows that

$$(U_\lambda(g)F)(X) = F(g^{-1}X)\chi_\lambda\left(s_\lambda(g^{-1}X)^{-1}g^{-1}s_\lambda(X)\right),$$

and the cyclic vector $F_{\lambda,e} \in K_\lambda$ corresponding to $f_{\lambda,e} \in \mathcal{H}_\lambda'$ defined by

$$f_{\lambda,e}(g) := \begin{cases} \chi_\lambda(g), & \text{if } g \in H_\lambda, \\ 0, & \text{otherwise}, \end{cases}$$

is a function (up to scalar factor) such that

$$F_{\lambda,e}(X) := \begin{cases} 1, & \text{if } X = H_\lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, for the present purpose, we only have to examine the following question (Q):

(Q) Let $G_\lambda \in \ell^2(\mathfrak{S}_\infty/H_\lambda)(\equiv K_\lambda)$ and $\int_{\mathbb{Z}^\infty} \|G_\lambda\|^2_{K_\lambda} d\mu_\beta(\lambda) < \infty$.

Suppose that

$$< G_\lambda, U_\lambda(g)F_{\lambda,e} >_{K_\lambda} = G_\lambda(gH_\lambda)\,\chi_\lambda\left(s_\lambda(H_\lambda)^{-1}g^{-1}s_\lambda(gH_\lambda)\right)$$

is a measurable function of λ, and

$$\int_{\mathbb{Z}^\infty} G_\lambda(gH_\lambda)\,\chi_\lambda\left(s_\lambda(H_\lambda)^{-1}g^{-1}s_\lambda(gH_\lambda)\right)\,d\mu_\beta(\lambda) = 0 \ \text{ for all } \ g \in \mathfrak{S}_\infty.$$

Then, does it implies that $G_\lambda = 0$ for μ-a.e. λ ?

The next theorem is a partial answer to the question (Q).

Theorem 2.5 *For $\beta = \{\beta_i\}_{i=-\infty}^{+\infty}$ with $\beta_0 = 0$, if we have either $\beta_i = \beta_j \neq 0$ or $\beta_{-i} = \beta_{-j} \neq 0$ for some different $i, j \in \mathbb{N}$, then (Q) is negative, so that Obata's disintegration of ϕ_β is not natural.*

Proof. Suppose that the first assumption holds. We take a transposition $p := (i, j)$ on $\mathbb{Z}$. This transposition acts from the left on $\mathbb{Z}^\infty$ such that $(p\lambda)_n = p(\lambda_n)$, where λ_n is the n-th component of $\lambda \in \mathbb{Z}^\infty$. Note that

$$p\mu_\beta = \mu_\beta, \quad H_{p\lambda} = H_\lambda \quad \text{and} \quad \chi_{p\lambda} = \chi_\lambda.$$

Define a function $G_\lambda \in \ell^2(\mathfrak{S}_\infty/H_\lambda)$ by

$$G_\lambda(X) := \psi_{\{\omega\in\mathbb{Z}^\infty | X = H_\omega\}}(\lambda) \cdot \left[\psi_{\{i\}}(\lambda_1) - \psi_{\{j\}}(\lambda_1)\right],$$

using the indicator function $\psi_{\{i\}}$ of the set $\{i\}$. Then,

$$\int_{\mathbb{Z}^\infty} \|G_\lambda\|^2_{K_\lambda}\, d\mu_\beta(\lambda) = 2\beta_i,$$

which is easily checked, and furthermore,

$$\begin{aligned} G_\lambda(gH_\lambda)\, \chi_\lambda\left(s_\lambda(H_\lambda)^{-1} g^{-1} s_\lambda(gH_\lambda)\right) &= \psi_{\{\omega\in\mathbb{Z}^\infty | gH_\lambda = H_\omega\}}(\lambda) \times \\ &\times \left[\psi_{\{i\}}(\lambda_1) - \psi_{\{j\}}(\lambda_1)\right] \chi_\lambda(g^{-1}). \end{aligned}$$

Therefore, the above function is measurable, and it has the opposite sign and the same absolute value according to the change of λ to $p\lambda$. This leads to

$$\int_{\mathbb{Z}^\infty} G_\lambda(gH_\lambda)\, \chi_\lambda\left(s_\lambda(H_\lambda)^{-1} g^{-1} s_\lambda(gH_\lambda)\right)\, d\mu_\beta(\lambda) = 0 \quad \text{for all} \quad g \in \mathfrak{S}_\infty.$$

While, we have $G_\lambda \neq 0$ for μ-a.e. λ, and this demonstrates the proof. The second case is similar. □

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