MSC 43A85

Poisson transforms for the complex hyperboloid [1]

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The representation U of the Lorentz group $G = \mathrm{SL}(2,\mathbb{C})$ by translations in polynomials on the complex hyperboloid $\mathcal{X}$ in $\mathbb{C}^3$ is the multiplicity free direct sum of finite dimensional representations. We write operators (Poisson transforms) intertwining finite dimensional representations in a standard realization with the representation U. Using a non-usual form of intertwining operators for principal series of G we write a differential form for Poisson transforms. Finally, we write explicitly spherical functions on $\mathcal{X}$.

Keywords: the Lorentz group, representations, distributions, hyperboloid, Poisson transforms, spherical functions

§ 1. Principal series and intertwining operators

The group $G = \mathrm{SL}(2,\mathbb{C})$ consists of complex 2×2 matrices:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \tag{1.1}$$

For such a matrix g, let $\widehat{g}$ denote the matrix obtained by permutations $\alpha \longleftrightarrow \delta$ and $\beta \longleftrightarrow \gamma$:

$$\widehat{g} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}.$$

The map $g \mapsto \widehat{g}$ is an involutive automorphism of the group G.

For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$, $a \in \mathbb{C}\backslash\{0\}$, we denote

$$z^{\lambda,k} = |z|^\lambda \left(\frac{z}{|z|}\right)^k.$$

Observe that the following differentiation formulae hold:

$$\frac{\partial}{\partial z}\, z^{\lambda,k} = \frac{\lambda+k}{2}\, z^{\lambda-1,k-1}, \quad \frac{\partial}{\partial \overline{z}}\, z^{\lambda,k} = \frac{\lambda-k}{2}\, z^{\lambda-1,k+1}.$$

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We also use the notation ("generalized" powers):

$$a^{(n)} = a(a-1)(a-2)\dots(a-n+1).$$

Define principal series representations of the group G. Let $\sigma \in \mathbb{C}$, $2m \in \mathbb{Z}$. Denote by $\mathcal{D}_{\sigma,m}$ the space of functions $f(z)$ in $C^\infty(\mathbb{C})$ such that "inverse" functions $\widehat{f}(z) = z^{2\sigma,2m} f(-1/z)$ also belong $C^\infty(\mathbb{C})$. The representation $T_{\sigma,m}$ of the principal series acts on $\mathcal{D}_{\sigma,m}$ by

$$\big(T_{\sigma,m}(g)f\big)(z) = f(z\cdot g)\,(\beta z+\delta)^{2\sigma,2m}\,, \quad z\cdot g = \frac{\alpha z+\gamma}{\beta z+\delta}\,.$$

The contragredient representation $\widehat{T}_{\sigma,m}$ is obtained from $T_{\sigma,m}$ by means of the involution $g \mapsto \widehat{g}$:

$$\widehat{T}_{\sigma,m}(g) = T_{\sigma,m}(\widehat{g}).$$

Representations $T_{\sigma,m}$ and $\widehat{T}_{\sigma,m}$ are equivalent. A bilinear form

$$\langle F, f\rangle_{\mathbb{C}} = \int_{\mathbb{C}} F(z)\, f(z)\, dx\, dy\,, \quad z = x+iy,$$

is invariant with respect to the pair $(T_{\sigma,m}, T_{-\sigma-2,-m})$.

Introduce an operator $A_{\sigma,m}$:

$$\left(A_{\sigma,m} f\right)(z) = \int_{\mathbb{C}} (1-zw)^{-2\sigma-4,-2m} f(w)\, du\, dv\,, \quad w = u+iv.$$

This operator intertwines $T_{\sigma,m}$ with $\widehat{T}_{-\sigma-2,-m}$ as well $\widehat{T}_{\sigma,m}$ with $T_{-\sigma-2,-m}$:

$$A_{\sigma,m} T_{\sigma,m}(g) = \widehat{T}_{-\sigma-2,-m}(g) A_{\sigma,m},$$

$$A_{\sigma,m} \widehat{T}_{\sigma,m}(g) = T_{-\sigma-2,-m}(g) A_{\sigma,m}.$$

[Notice that often one takes an intertwining operator with the kernel $(z-w)^{-2\sigma-4,-2m}$. It intertwines $T_{\sigma,m}$ with $T_{-\sigma-2,-m}$. But for our purposes (we want to construct polynomial quantization and to study finite dimensional analyse on a complex hyperboloid), just introduced operator $A_{\sigma,m}$ is much more convenient.]

The composition of operators $A_{-\sigma-2,-m}$ and $A_{\sigma,m}$ is a scalar operator, i. e. an operator of multiplcation by a number:

$$A_{-\sigma-2,-m} A_{\sigma,m} = \omega_{0,0}(\sigma, m) E,$$

where

$$\omega_{0,0}(\sigma, m) = -(-1)^{2m} \frac{\pi^2}{(\sigma+1)^2 - m^2}\,.$$

Let $\mathfrak{g}$ be the Lie algebra of the group G. It acts on $\mathcal{D}_{\sigma,m}$ by means of some first order differential operators. These operators give rise to a representation of the Lie algebra $\mathfrak{g}$ and its universal enveloping algebra not only on $\mathcal{D}_{\sigma,m}$, but also on other spaces, for example, on the space $C^\infty(\mathbb{C})$, on the space $\mathrm{Pol}\,(\mathbb{C})$ of polynomials on $\mathbb{C}$, on the space $\mathcal{D}'(\mathbb{C})$ of distributions on $\mathbb{C}$, on the space $\mathcal{D}'_0(\mathbb{C})$ distributions on $\mathbb{C}$ concentrated at zero.

The space $\mathcal{D}'_0(\mathbb{C})$ consists of linear combinations of the delta function $\delta(z,\overline{z})$ and its derivatives

$$\delta^{(k,l)}(z,\overline{z}) = \frac{\partial^{k+l}}{\partial z^k\,\partial \overline{z}^l}\,\delta(z,\overline{z}).$$

The spaces $\mathrm{Pol}\,(\mathbb{C})$ и $\mathcal{D}'_0(\mathbb{C})$ are Verma modules with respect to $T_{\sigma,m}$.

The intertwining operator $A_{\sigma,m}$ carries out the basis $z^k\overline{z}^l$ in $\mathrm{Pol}\,(\mathbb{C})$ to the basis $\delta^{(k,l)}(z,\overline{z})$ in $\mathcal{D}'_0(\mathbb{C})$ and back (with factors):

$$A_{\sigma,m}\,\left(z^k\overline{z}^l\right) = \omega_{k,l}(\sigma,m)\cdot\delta^{(k,l)}(z,\overline{z})\,, \tag{1.2}$$

$$A_{-\sigma-2,-m}\,\left(\delta^{(k,l)}(z,\overline{z})\right) = (\sigma+m)^{(k)}(\sigma-m)^{(l)}\cdot z^k\overline{z}^l,$$

where

$$\omega_{k,l}(\sigma,m) = -(-1)^{2m}\frac{\pi^2}{(\sigma+1+m)^{(k+1)}(\sigma+1-m)^{(l+1)}}\,. \tag{1.3}$$

§ 2. Finite dimensional representations

Let $2k,\ 2l \in \mathbb{N} = \{0,1,2,\ldots\}$. Let $V_{k,l}$ be the space of polynomials $\varphi(z,\overline{z})$ in two variables z and $\overline{z}$ of degrees $\leqslant 2k$ and $\leqslant 2l$ in variables z and $\overline{z}$ respectively. On this space the representation $\pi_{k,l}$ of G acts by

$$(\pi_{k,l}(g)\varphi)(z,\overline{z}) = \varphi(z\cdot g, \overline{z\cdot g})(\beta z+\delta)^{2k}(\overline{\beta}\overline{z}+\overline{\delta})^{2l}.$$

A basis in $V_{k,l}$ consists of monomials $z^p\overline{z}^q$, where $0\leqslant p\leqslant 2k$, $0\leqslant q\leqslant 2l$, so that $\dim V_{k,l} = (2k+1)(2l+1)$. All representations $\pi_{k,l}$ are irreducible. Conversely, any irreducible finite dimensional representation is equivalent to one of $\pi_{k,l}$. In particular, the contragredient representation $\widehat{\pi}_{k,l}$ defined by $\widehat{\pi}_{k,l}(g) = \pi_{k,l}(\widehat{g})$ is equivalent to $\pi_{k,l}$.

The representation $\pi_{k,l}$ preserves the following bilinear form $B_{k,l}(\psi,\varphi)$ on $V_{k,l}$: at basis elements it is given by

$$B_{k,l}(z^r\overline{z}^s, z^p\overline{z}^q) = (-1)^{p+q}\binom{2k}{p}^{-1}\binom{2l}{q}^{-1}\delta_{p,2k-r}\delta_{q,2l-s},$$

$\delta_{i,j}$ being the Kronecker delta. Such a form is unique up to a factor. Parallel with it we consider on $V_{k,l}$ the bilinear form $B'_{k,l}(\psi,\varphi) = B_{k,l}(\psi,\widehat{\varphi})$, so that

$$B'_{k,l}(z^r\overline{z}^s, z^p\overline{z}^q) = (-1)^{p+q}\binom{2k}{p}^{-1}\binom{2l}{q}^{-1}\delta_{p,r}\delta_{q,s}. \tag{2.1}$$

The form $B'_{k,l}$ is invariant with respect to the pair $(\widehat{\pi}_{k,l},\pi_{k,l})$, i. e.

$$B'_{k,l}(\psi,\pi_{k,l}(g)\varphi) = B'_{k,l}(\widehat{\pi}_{k,l}(g^{-1})\psi,\varphi).$$

For $\sigma = k + l$, $m = k - l$, the representation $T_{\sigma,m} = T_{k+l,k-l}$ is reducible, it has an invariant finite dimensional irreducible subspace $V_{k,l}$. The module $\mathcal{D}_0'(\mathbb{C})$ with respect to $T_{-k-l-2,-k+l}$ has a submodule invariant with respect to $\mathfrak{g}$. The quotient module is equivalent to $V_{k,l}$.

We can write the form $B'_{k,l}$ by means of the intertwining operator $A_{k+l,k-l}$ (which is defined just on $V_{k,l}$):

$$B'_{k,l}(\psi, \varphi) = -\frac{1}{\pi^2}\,(2k+1)\,(2l+1)\,\langle A_{k+l,k-l}\psi, \varphi\rangle_{\mathbb{C}}\,, \tag{2.2}$$

it follows from (1.2), (1.3) and (2.1).

§ 3. Hyperboloid

Introduce in $\mathbb{C}^3$ a bilinear form

$$[x, y] = -x_1y_1 + x_2y_2 + x_3y_3.$$

Denote by $\mathcal{X}$ the hyperboloid $[x, x] = 1$. The manifold $\mathcal{X}$ can be realized as a set of matrices

$$x = \frac{1}{2}\begin{pmatrix} 1 - x_3 & x_2 - x_1 \\ x_2 + x_1 & 1 + x_3 \end{pmatrix}$$

with $\det x = 1$. The group G acts on the space $\mathrm{Mat}(2, \mathbb{C})$ as follows: $x \to g^{-1}xg$. On the hyperboloid $\mathcal{X}$ it acts transitively. The stabilizer of the point

$$x^0 = (0, 0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is the diagonal subgroup H of G, it consists of matrices

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad \alpha\delta = 1.$$

An action of G on functions f on $\mathcal{X}$ by translations is denoted by U:

$$(U(g)f)(x) = f(g^{-1}xg), \quad g \in G.$$

Introduce on $\mathcal{X}$ *horospherical coordinates* ξ, η:

$$x = N^{-1}(\xi + \eta,\, \xi - \eta,\, 1 + \xi\eta), \quad N = 1 - \xi\eta,$$

in the matrix form we have:

$$x = \frac{1}{N}\begin{pmatrix} -\xi\eta & -\eta \\ \xi & 1 \end{pmatrix},$$

These coordinates are defined on $\mathcal{X}$ except $x_3 = -1$. The action $x \mapsto g^{-1}xg$ of G is given by a linear-fraction transformaton in each variable ξ and η separately:

$$\xi \to \widetilde{\xi} = \xi \cdot g, \quad \eta \to \widehat{\eta} = \eta \cdot \widehat{g},$$

so that $(U(g)f)(\xi,\eta) = f(\widetilde{\xi},\widehat{\eta})$. The initial point x^0 has coordinates $\xi = 0$, $\eta = 0$. An element g, see (1.1), moves x^0 to a point x with coordinates

$$\xi = \frac{\gamma}{\delta}, \quad \eta = \frac{\beta}{\alpha}, \tag{3.1}$$

so that $N = 1/\alpha\delta$.

There are two Laplace operators Δ и $\overline{\Delta}$ on $\mathcal{X}$:

$$\Delta = N^2 \frac{\partial^2}{\partial\xi\partial\eta}, \quad \overline{\Delta} = \overline{N}^2 \frac{\partial^2}{\partial\bar{\xi}\partial\overline{\eta}},$$

they are generators in the algebra of G-invariant differential operators on $\mathcal{X}$.

Let us denote by $\mathcal{A}$ and $\overline{\mathcal{A}}$ spaces of analytic and antianalytic polynomials in $\mathbb{C}^3$ respectively. A polynomial f in $\mathcal{A}$ is called *harmonic* (with respect to the form $[x,y]$) if

$$\left(-\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) f = 0.$$

Let us denote by $\mathcal{H}$ the space of harmonic polynomials and by $\overline{\mathcal{H}}$ the corresponding subspace in $\overline{\mathcal{A}}$. Let $\mathcal{H}_{k,l}$ be the subspace in $\mathcal{H} \otimes \overline{\mathcal{H}}$ consisting of homogeneous polynomials of degree k in x_1, x_2, x_3 and degree l in $\overline{x}_1, \overline{x}_2, \overline{x}_3$. We denote restrictions of spaces $\mathcal{H} \otimes \overline{\mathcal{H}}$ and $\mathcal{H}_{k,l}$ to $\mathcal{X}$ by $(\mathcal{H} \otimes \overline{\mathcal{H}})(\mathcal{X})$ and $\mathcal{H}_{k,l}(\mathcal{X})$ respectively. This restricting map is one-to-one. Notice that $(\mathcal{H} \otimes \overline{\mathcal{H}})(\mathcal{X})$ coincides with the space of restrictions to $\mathcal{X}$ of *all* polynomials on $\mathbb{C}^3$. The space $\mathcal{H}_{k,l}(\mathcal{X})$ is invariant and irreducible with respect to U, the corresponding representation is equivalent to $\pi_{k,l}$, see below. Polynomials in this space are eigenfunctions for Laplace operators:

$$\Delta f = k(k+1)f, \quad \overline{\Delta} f = l(l+1)f.$$

§ 4. Poisson transforms, spherical functions

For representations $\pi_{k,l}$ and $\widehat{\pi}_{k,l}$, an invariant with respect to H in the space $V_{k,l}$ exists for *integer* k and l only, it is the monomial

$$\theta_{k,l}(z,\overline{z}) = z^k \overline{z}^l.$$

It is unique up to a factor. It gives rise to an intertwining operator mapping $V_{k,l}$ to functions on $\mathcal{X}$, we call it the *Poisson transform* $\mathcal{P}_{k,l}$. This transform assigns to a polynomial $\varphi(z,\overline{z})$ in $V_{k,l}$ the following function $(\mathcal{P}_{k,l}\varphi)(x)$ on $\mathcal{X}$:

$$\begin{aligned}(\mathcal{P}_{k,l}\varphi)(x) &= B_{k,l}(\pi(g^{-1})\theta_{k,l}, \varphi) \\ &= B_{k,l}(\theta_{k,l}, \pi(g)\varphi),\end{aligned} \tag{4.1}$$

where g is an element in G such that $g^{-1}x^0g = x$. It intertwines $\pi_{k,l}$ and U:

$$\mathcal{P}_{k,l}\,\pi_{k,l}(g) = U(g)\,\mathcal{P}_{k,l}.$$

The forms $B_{k,l}(\psi,\varphi)$ and $B'_{k,l}(\psi,\varphi)$ coincide for $\psi = \theta_{k,l}$:

$$B_{k,l}(\theta_{k,l},\varphi) = B'_{k,l}(\theta_{k,l},\varphi).$$

Therefore, we can rewrite (4.1) as

$$\begin{aligned}(\mathcal{P}_{k,l}\varphi)(x) &= B'_{k,l}(\widehat{\pi}(g^{-1})\theta_{k,l},\varphi)\\ &= B'_{k,l}(\theta_{k,l},\pi(g)\varphi),\end{aligned}$$

The latter formula, together with (2.2), (1.2), (1.3), gives a *differential* form for the Poisson transform:

$$\begin{aligned}(\mathcal{P}_{k,l}\varphi)(x) &= c(k,l)\int_{\mathbb{C}}\delta^{(k,l)}(z,\overline{z})\Big(T_{k+l,k-l}(g)\varphi(z,\overline{z})\Big)\,dx\,dy\\ &= c(k,l)\cdot(-1)^{k+l}\frac{\partial^{k+l}}{\partial z^k\,\partial\overline{z}^l}\bigg|_{z=0}\varphi\left(z\cdot g,\overline{z\cdot g}\right)\,(\beta z+\delta)^{2k}\,(\overline{\beta}\overline{z}+\overline{\delta})^{2l},\end{aligned}$$

where $c(k,l) = k!\,l!/(2k)!\,(2l)!$. We get $(\mathcal{P}_{k,l}\varphi)(x)$ as a function of horospherical coordinates ξ,η, see (3.1). This function is just a polynomial on $\mathcal{X}$ lying in $\mathcal{H}_{k,l}(\mathcal{X})$. Therefore, the Poisson transform $\mathcal{P}_{k,l}$ maps $\pi_{k,l}$ isomorphically onto $\mathcal{H}_{k,l}(\mathcal{X})$.

In particular, H-invariant $\theta_{k,l}$ goes to a H-invariant $\Psi_{k,l}$ in $\mathcal{H}_{k,l}(\mathcal{X})$ (a *spherical function*):

$$\begin{aligned}\Psi_{k,l}(x) &= (-1)^{k+l}\binom{2k}{k}^{-1}\binom{2l}{l}^{-1}N^{-k}\sum_{j=0}^{k}\binom{k}{j}^2(\xi\eta)^j\times\\ &\times\ \overline{N}^{-l}\sum_{i=0}^{l}\binom{l}{i}^2(\overline{\xi}\overline{\eta})^i\\ &= (-1)^{k+l}\binom{2k}{k}^{-1}\binom{2l}{l}^{-1}P_k(x_3)\,P_l(\overline{x}_3),\end{aligned}$$

where $P_m(t)$ is the Legendre polynomial.