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Harmonic analysis on the Lobachevsky–Galilei plane [1]

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Harmonic analysis on the Lobachevsky–Galilei plane is constructed
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§ 1. Lobachevsky–Galilei plane

Let Λ be an algebra over $\mathbb{R}$ of dimension 2 consisting of elements $z = x + iy$, $x, y \in \mathbb{R}$ with relation $i^2 = 0$ (the algebra of *dual numbers*). It is not a field: pure imaginary numbers iy are zero divisors. For $z = x + iy$, the conjugate number is $\overline{z} = x - iy$.

The Lobachevsky–Galilei plane $\mathcal{L}$ is a domain on the plane Λ, defined by $z\overline{z} < 1$. It is a vertical strip bounded by lines $x = \pm 1$. Denote the line $x = 1$ by Γ. The group G of translations of the Lobachevsky–Galilei plane $\mathcal{L}$ consists of linear-fractional transformations

$$z \mapsto z \cdot g = \frac{az + \overline{b}}{bz + \overline{a}}, \quad a\overline{a} - b\overline{b} = 1, \quad a, b \in \Lambda. \tag{1.1}$$

It preserves the measure

$$d\sigma(z) = \frac{dx\, dy}{(1 - x^2)^2}.$$

Matrices

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad a\overline{a} - b\overline{b} = 1,$$

occuring in (1.1), form the group $\mathrm{SU}(1,1;\Lambda)$. Denote $a = \alpha + ip$, $b = \beta + iq$. The condition $a\overline{a} - b\overline{b} = 1$ is equivalent to $\alpha^2 - \beta^2 = 1$, so that $\alpha^2 \geqslant 1$, therefore $\alpha \geqslant 1$ or $\alpha \leqslant -1$. Thus, the group $\mathrm{SU}(1,1;\Lambda)$ consists of two connected parts. The connected component of the identity is isomorphic just to the group G. This component consists

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of matrices g with $\alpha \geqslant 1$. For this component we retain the notation G. We can write parameters α and β of the matrix $g \in G$ as $\alpha = \operatorname{ch} t$ и $\beta = \operatorname{sh} t$, where $t \in \mathbb{R}$. Therefore, any matrix $g \in G$ can be written as follows

$$g = g(t) + ic(p, q), \tag{1.2}$$

where

$$g(t) = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix}, \quad c(p, q) = \begin{pmatrix} p & q \\ -q & -p \end{pmatrix}.$$

The stabilizer of the point $z = 0$ is the subgroup K consisting of diagonal matrices:

$$k = \begin{pmatrix} 1 + ip & 0 \\ 0 & 1 - ip \end{pmatrix}, \tag{1.3}$$

so that

$$\mathcal{L} = G/K.$$

Let $(f, h)_{\mathcal{L}}$ be the inner product in the space $L^2(\mathcal{L}, d\sigma)$:

$$(f, h)_{\mathcal{L}} = \int_{\mathcal{L}} f(z)\, \overline{h(z)}\, d\sigma(z).$$

The quasiregular representation U of G acts on this space: $(U(g)f)(z) = f(z \cdot g)$.

§ 2. Representations of the group G

In this section we describe two series of representations of the group G induced by characters (one-dimensional representations) of "parabolic" subgroups P_0 и P_∞. The first one P_0 is the stabilizer of the point $\gamma_0 = 1$ in Γ, the second one P_∞ is obtained by a limit passage from the stabilizer of the point $\gamma_y = 1 + iy$ when $y \to \infty$.

The subgroup P_0 consists of matrices (1.2) with $p = q$, i. e. matrices $h(t, q) = g(t) + ic(q, q)$. Its character ω_λ is defined by a complex number λ:

$$\omega_\lambda(h) = e^{\lambda t} = (\alpha + \beta)^\lambda.$$

The set G/P_0 can be identified with the line Γ: to a point γ_y one assigns the diagonal matrix (1.3) with $p = y/2$. The representation T_λ of G induced by ω_λ acts in functions $\varphi(\gamma)$ in $\mathcal{D}(\Gamma)$ by

$$T_\lambda(g)\varphi(\gamma) = \varphi\left(\gamma \cdot g\right)(\alpha + \beta)^\lambda, \quad \lambda \in \mathbb{C}.$$

The stabilizer P_y of the point γ_y consists of matrices (1.2) such that $p = q + y \operatorname{sh} t$. The subgroups P_y and P_0 are isomoephic, so representations induced by characters these subgroups are equivalent.

Let us find the limit P_∞ of P_y when $y \to \infty$. We consider parameters t and q depending on y: $t = t_y$, $q = q_y$, in such way that the following limits exist: $\lim t_y = t$,

$\lim q_y = q$, $\lim(q_y + y\,\mathrm{sh} t_y) = p$. It gives $t = 0$. Therefore, the subgroup P_∞ consists of matrices

$$h = h(u, v) = E + ic(u, v), \tag{2.1}$$

where E is the identity matrix of the second order. This subgroup is a commutative normal subgroup of G, isomorphic to $\mathbb{R}^2$, so that its characters are

$$\omega_{\lambda,\mu}(h) = e^{\lambda u} e^{\mu v},$$

where $\lambda, \mu \in \mathbb{C}$ and h is the matrix (2.1). Any matrix (1.2) can be written as

$$g = h(u, v)\, g(t), \tag{2.2}$$

where

$$h(u, v) = E + ic(p, q) g(t)^{-1},$$

so that

$$u = p\ \mathrm{ch}\, t - q\ \mathrm{sh}\, t, \quad v = -p\ \mathrm{sh}\, t + q\ \mathrm{ch}\, t.$$

So we can identify G/P_∞ with the subgroup of matrices $g(t)$ and hence with $\mathbb{R}$. The representation $T_{\lambda,\mu}$ of G induced by $\omega_{\lambda,\mu}$ acts in functions $\varphi(s)$ in $\mathcal{D}(\mathbb{R})$ by:

$$T_{\lambda,\mu}(g)\varphi(s) = \varphi(\widetilde{s})\, \omega_{\lambda,\mu}(\widetilde{h}),$$

where $\widetilde{s}$ and $\widetilde{h}$ are obtained if we decompose $g(s)\, g$ in accordance with (2.2): $g(s)\, g = \widetilde{h}\, g(\widetilde{s})$. Let us write g also as (2.2) then

$$g(s)\, g = \{E + ig(s)c(u, v)g(s)^{-1}\} \cdot g(s + t),$$

so that $\widetilde{s} = s + t$, $\widetilde{h}(u, v) = E + i\, c(\widetilde{u}, \widetilde{v})$, where

$$\begin{aligned} c(\widetilde{u}, \widetilde{v}) &= g(s)c(u, v)g(s)^{-1} \\ &= g(s)c(p, q)g(s + t)^{-1}, \end{aligned}$$

therefore,

$$\widetilde{u} = p\ \mathrm{ch}(2s + t) - q\ \mathrm{sh}(2s + t), \tag{2.3}$$

$$\widetilde{v} = -p\ \mathrm{sh}(2s + t) + q\ \mathrm{ch}(2s + t). \tag{2.4}$$

Finally for $g = g(t) + ic(p, q)$ we get

$$\left(T_{\lambda,\mu}(g)\, \varphi\right)(s) = \varphi(s + t)\, e^{\lambda \widetilde{u} + \mu \widetilde{v}}, \tag{2.5}$$

with $\widetilde{u}$ and $\widetilde{v}$ given by (2.3) and (2.4).

A Hermitian form (the inner product from $L^2(\mathbb{R}, ds)$)

$$\langle \psi, \varphi \rangle = \int_{-\infty}^{\infty} \psi(s)\overline{\varphi(s)}\, ds$$

is invariant with respect to the pair $(T_{\lambda,\mu}, T_{-\overline{\lambda},-\overline{\mu}})$, i. e.

$$\langle T_{\lambda,\mu}(g)\psi, \varphi\rangle = \langle \psi, T_{-\overline{\lambda},-\overline{\mu}}(g^{-1})\varphi\rangle, \tag{2.6}$$

so that $T_{\lambda,\mu}$ is unitarizable for pure imaginary λ, μ.

Using (2.6) we can extend $T_{\lambda,\mu}$ to the space $\mathcal{D}'(\mathbb{R})$ of distributions ψ on $\mathbb{R}$.

§ 3. Poisson and Fourier transforms, spherical functions

We need the second series from § 2.

Theorem 3.1 *A non-trivial space of K-invariants in $\mathcal{D}'(\mathbb{R})$ under $T_{\lambda,\mu}$ exists provided that $\lambda = r\mu$, where $-1 < r < 1$; if so let us set $r = \operatorname{th} 2\tau$, $\tau \in \mathbb{R}$, so that*

$$\lambda = \operatorname{th} 2\tau \cdot \mu, \quad \tau \in \mathbb{R}. \tag{3.1}$$

This space is one-dimensional, a basis function θ is the delta function:

$$\theta(s) = \delta(s - \tau).$$

Proof. Let a function $\theta(s)$ is K-invariant under $T_{\lambda,\mu}$. By (1.3) and (2.5), (2.3), (2.4) it means:

$$e^{p(\lambda \operatorname{ch} 2s - \mu \operatorname{sh} 2s)}\,\theta(s) = \theta(s)$$

for all $p \in \mathbb{R}$. This is equivalent to a condition that is obtained by differentiation with respect to p at zero: $(\lambda \operatorname{ch} 2s - \mu \operatorname{sh} 2s)\,\theta(s) = 0$, or $(\lambda - \mu \operatorname{th} 2s)\,\theta(s) = 0$. The factor in front of $\theta(s)$ has to vanish at some point $s = \tau$. It gives (3.1). □

The representation $T_{\lambda,\mu}$ with condition (3.1) is equivalent to that with $\lambda = 0$. Indeed, the translation operator C: $(C\varphi)(s) = \varphi(s + \tau)$, intertwines $T_{\lambda,\mu}$ with $T_{0,\nu}$, where $\nu = \mu/\operatorname{ch} 2\tau$. So we can take $\lambda = 0$ from the beginning. Then Theorem 3.1 claims that the representation $T_{0,\mu}$, $\mu \in \mathbb{C}$, has a K-invariant $\theta(s) = \delta(s)$ unique up to a factor.

The K-invariant $\theta(s) = \delta(s)$ gives rise to a Poisson kernel

$$P_\mu(z,s) = \left(T_{0,\mu}(g^{-1})\theta\right)(s), \quad z \in \mathcal{L}, \quad s \in \mathbb{R},$$

where g is an element in G moving 0 to z, for example, the matrix

$$g_z = \frac{1}{\sqrt{1-x^2}} \begin{pmatrix} 1 & \overline{z} \\ z & 1 \end{pmatrix}, \quad z = x + iy.$$

We get

$$\begin{aligned} P_\mu(z,s) &= \delta(\xi - s) \exp\left\{\mu \frac{y}{1-x^2}\right\} \\ &= (1-x^2)\,\delta(x-c) \exp\left\{\mu \frac{y}{1-x^2}\right\}, \end{aligned}$$

where $x = \operatorname{th}\xi$, $c = \operatorname{th} s$. This kernel gives rise to two transforms: the *Poisson transform* $P_\mu : \mathcal{D}(\mathbb{R}) \to C^\infty(\mathcal{L})$ and the *Fourier transform* $F_\mu : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathbb{R})$ defined respectively by:

$$\begin{aligned}(P_\mu \varphi)(z) &= \int_{-\infty}^{\infty} P_\mu(z,s)\,\varphi(s)\,ds = \varphi(\xi)\,\exp\left\{\mu\frac{y}{1-x^2}\right\}, \qquad (3.2)\\ (F_\mu f)(s) &= \int_{\mathcal{L}} P_\mu(z,s)\,f(z)\,d\sigma(z)\\ &= \frac{1}{1-c^2}\int_{-\infty}^{\infty} f(c+iy)\,\exp\left\{\mu\frac{y}{1-c^2}\right\}dy,\ \ c=\operatorname{th} s.\end{aligned}$$

They intertwine $T_{0,-\mu}$ with U and U with $T_{0,\mu}$, respectively. These transforms are conjugate to each other:

$$(P_\mu \varphi, f)_{\mathcal{L}} = \langle \varphi, F_{\overline{\mu}} f\rangle.$$

The *spherical function* Ψ_μ is defined as the Poisson image of the K-invariant θ:

$$\Psi_\mu = P_\mu\,\theta.$$

It follows from (3.2) that

$$\Psi_\mu(z) = \delta(x)\,e^{\mu y}.$$

§ 4. Decomposition of the quasiregular representation

Theorem 4.1 *The quasiregular representation U of the group G in $L^2(\mathcal{L}, d\sigma)$ decomposes in the direct integral of representations $T_{0,i\rho}$, $\rho \in \mathbb{R}$, with multiplicity one as follows (here $i = \sqrt{-1} \in \mathbb{C}$, the complex number). Let us assign to a function $f \in \mathcal{D}(\mathcal{L})$ the family of its Fourier components Фурье $F_{i\rho} f$, $\rho \in \mathbb{R}$. This correspondence is G-equivariant. There are an inverse formula:*

$$f = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{-i\rho}\,F_{i\rho}\,f\,d\rho, \qquad (4.1)$$

and the Plancherel formula:

$$(f,h)_{\mathcal{L}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle F_{i\rho}\,f, F_{i\rho}\,h\rangle\,d\rho. \qquad (4.2)$$

Therefore the map $f \mapsto \{F_{i\rho} f\}$ can be extended to the whole space $L^2(\mathcal{L}, d\sigma)$.

Formulas (4.1), (4.2) are obtained from corresponding formulas for the classical Fourier transform by the change $\rho = (1-x^2)\eta$.

These formulas can be united by the formula decomposing the delta function $\delta(z)$ concentrated at $z = 0$ into spherical functions:

$$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_{i\rho}(z)\,d\rho.$$