# Type of von Neumann algebras generated by regular representations of infinite dimensional groups

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Let us consider ( see [1]) an analogue of the right and the left regular representation of the group $B_0^\infty = \lim\limits_{\to} B(n, \mathbf{R})$ of finite upper-triangular matrices of infinite order: $T^{R,b}$, $T^{L,b} : B_0^\infty \to U(H_b = L_2(B^\infty, d\mu_b))$,

$$(T_t^{R,b} f)(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2} f(xt),$$
$$(T_s^{L,b} f)(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2} f(s^{-1}x),$$

corresponding to a $B_0^\infty-$ quasi-invariant measure $\mu_b$ on the group $B^\infty$ of all upper triangular matrices, where $\mu_b$ is defined as follows:

$$d\mu_b(x) = \otimes_{k<n}(b_{kn}/\pi)^{1/2} \exp(-b_{kn}x_{kn}^2)dx_{kn} = \otimes_{k<n} d\mu_{b_{kn}}(x_{kn})$$

and $b = (b_{kn})_{k<n}$ is a set of positive numbers.

Let us denote by $R$ and $L$ the right and the left action of the group $B_0^\infty$ on $B^\infty : R_s(t) = ts^{-1}, \ \ L_s(t) = st, s \in B_0^\infty, t \in B^\infty$.

Let $\mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^\infty)''$ and $\mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^\infty)''$.

**Theorem 1** [1] *The von Neumann algebra $\mathfrak{A}^{R,b}$ is type $\mathrm{I}_\infty$ factor if and only if $\mu_b^{L_s} \perp \mu_b \ \forall s \in B_0^\infty$.*

Let now assume that $\mu_b(x^{-1}) \sim \mu_b(x)$ then all left actions are admissible for measure $\mu_b$ i.e. $\mu_b^{L_s} \sim \mu_b \ \forall s \in B_0^\infty$. In this case the canonical conjugation $J$ is $(Jf)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2}\overline{f}(x^{-1})$ and we have $JT_t^{R,b}J = T_t^{L,b}, \ t \in B_0^\infty$.

**Theorem 2** *If $\mu_b(x^{-1}) \sim \mu_b(x)$ then $(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}$.*

**Theorem 3** *If $\mu_b(x^{-1}) \sim \mu_b(x)$ then the von Neumann algebra $\mathfrak{A}^{L,b}$ is factor.*

We shall prove that $M \bigcap M' = \{\lambda \mathbf{I} \mid \lambda \in \mathbf{C}^1\}$ where $M = \mathfrak{A}^{L,b}$. Since $M' = (\mathfrak{A}^{L,b})' = \mathfrak{A}^{R,b}$ it is equivalent to the fact that the representation

$$B_0^\infty \times B_0^\infty \ni (t.\, s) \to T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible.

Let us denote $T(G) = \{T_t \mid t \in G\}$,

$$B(p) = B(p, \mathrm{R}), \ B^p = \{t \in B^\infty \mid t = I + \sum_{k<n, k\le p} t_{kn}E_{kn}\}, \ B_0^p = B^p \bigcap B_0^\infty,$$

$$B_p = \{t \in B^\infty \mid t = I + \sum_{p<k<n} t_{kn}E_{kn}\},$$

$$\mu_b^p(x) = \otimes_{k<n, k\le p} \mu_{b_{kn}}(x_{kn}),$$

$$\mu_{b,p}(x) = \otimes_{p<k<n} \mu_{b_{kn}}(x_{kn}),$$

$$H_b^p = L_2(B^p, d\mu_b^p), \ H_{b,p} = L_2(B_p, d\mu_{b,p}),$$

then $H_b = H_b^p \otimes H_{b,p}$. Since $B_0^\infty = \bigcup_{p=2}^\infty B(p) = \bigcup_{p=2}^\infty B^p$ we have

$$M\bigcap M' = \mathfrak{A}^{L,b}\bigcap(\mathfrak{A}^{L,b})' = (\mathfrak{A}^{R,b}\bigcup\mathfrak{A}^{L,b})' =$$

$$(T^{R,b}(B_0^\infty)\bigcup T^{L,b}(B_0^\infty))' = (\bigcup_{p,r=2}^{\infty}(T^{R,b}(B^p)\bigcup T^{L,b}(B(r))))' \subset$$

$$(\bigcup_{p=2}^{\infty}(T^{R,b}(B^p)\bigcup T^{L,b}(B(p))) \otimes \mathbf{I}_{H_{b,p}})' =$$

$$\bigcap_{p=2}^{\infty}(T^{R,b}(B^p)\bigcup T^{L,b}(B(p)))' \otimes B(H_{b,p}) =$$

$$\bigcap_{p=2}^{\infty}((T^{L,b}(B^p))''\bigcap(T^{L,b}(B(p)))') \otimes B(H_{b,p}) =$$

$$\bigcap_{p=2}^{\infty}(T^{L,b}(\mathbf{Z}(B_0^p)))'' \otimes B(H_{b,p}) =$$

$$\{\lambda\mathbf{I} \mid \lambda \in \mathbf{C}^1\},$$

where $\mathbf{Z}(B_0^p)$ is the center of the group $B_0^p$ :

$$\mathbf{Z}(B_0^p) = \{t \in B_0^\infty \mid t = I + \sum_{n\geq p} t_{1n}E_{1n}\}.$$

# References

[1] A.V.Kosyak, Criteria for irreducibility and equivalence of regular Gaussian representations of group of finite upper-triangular matrices of infinite order, *Selecta. Math. Soviet.***11** (1992), 241-291.