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Indecomposable characters of the group of rational rearrangements of the segment [1]

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Let R be the group of rational rearrangements of the half-interval $[0,1)$. We present a countable family of characters of the group R. These characters are indecomposable

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§ 1. Introduction

Consider the half-interval $[0,1)$ (below it is called "the segment") equipped with the Lebesgue measure μ, and all automorphisms of this measure space such that the action of each of them is a "rearrangement" of a finite number of half-intervals with rational endpoints that form a partition of $[0,1)$.

The formal definition of these automorphisms is as follows.

Definition 11 A bijection $g\colon [0,1) \to [0,1)$ is called a *rational rearrangement of the segment if there exist a number $n \in \mathbb{N}$, a permutation $u \in \mathrm{S}_n$, and a partition $\{[x_0,x_1),\dots,[x_{n-1},x_n)\}$ of $[0,1)$ into n half-intervals with rational endpoints such that $g(x) = x - x_i + x_{u(i)}$ for every $i \in \{0,\dots,n-1\} = \underline{n}$ and $x \in [x_i,x_{i+1})$ (here and below the group S_n is defined as the group of all bijections of the set $\underline{n}$).*

It is easy to show that the set of all rational rearrangements is a dense subgroup in the automorphism group of the segment as a measure space; denote this subgroup by R. This group can also be described algebraically in terms of the symmetric groups and the operations on them defined below.

Definition 12 The *direct product* $u \sqcap v \in \mathrm{S}_{lm}$ of permutations $u \in \mathrm{S}_l$ and $v \in \mathrm{S}_m$ (where $l,m \in \mathbb{N}$) is the permutation defined for any $z \in \underline{lm}$ by the formula

$$u \sqcap v(z) = m\,u(z \operatorname{div} m) + v(z \bmod m).$$

Definition 13 The *periodic embedding* of the group S_l to the group S_{lm} is the map that sends any permutation $u \in \mathrm{S}_l$ to the permutation $u \sqcap \mathrm{id}_{\underline{m}} \in \mathrm{S}_{lm}$.

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It can be proved directly that the groups S_n, $n \in \mathbb{N}$, form an inductive family of groups with respect to the periodic embeddings and the ordering of the index set $\mathbb{N}$ by the divisibility relation.

Theorem 1 *The group* R *of rational rearrangements of the segment is the inductive limit of the symmetric groups* S_n, $n \in \mathbb{N}$, *with respect to the periodic embeddings.*

Proof. The group S_n is embedded into the group R as follows: a permutation $u \in \mathrm{S}_n$ goes to the rearrangement $g \in \mathrm{R}$ that moves the half-intervals $[k/n, (k+1)/n)$, $k = 0, 1, \ldots, n-1$, according to the action of u; hence

$$g(x) = \frac{u(\lfloor xn \rfloor) + \{xn\}}{n}, \quad x \in [0,1). \tag{1}$$

Any partition of $[0,1)$ into half-intervals with rational endpoints can be refined to a partition into half-intervals of the same length, so that any element of R can be obtained as above. Finally, it is easy to check that the constructed embeddings are compatible with the periodic embeddings of the groups S_n, $n \in \mathbb{N}$, and this completes the proof. □

Further, we study certain aspects of the representation theory of the group R and, in particular, its indecomposable characters. This group is an inductive limit of symmetric groups, so we can apply to it the general methods of the representation theory of inductive limits of finite groups, which are stated in the papers [6], [7]; these papers are fundamental for us. The analogous approach is possible for other examples of groups of the same type, such as the infinite symmetric group S_∞ (see [4], [5], [7]) and various groups of matrices over countable fields of finite characteristic (see [2], [3]). The study of the representation theory of the group R was, apparently, first started in the paper [1]. This paper also contains detailed proofs of certain statements used in the present renewed and abridged account.

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§ 2. Preliminaries on characters and the K_0-functor

In this section, we formulate some general definitions and facts related to the representation theory of groups that are inductive limits of finite groups; then we apply it to the group R.

Let a group G be an inductive limit of finite groups G_i, $i \in I$, where I is an at most countable directed set. Under this general assumption, we can define the following main objects of our study.

Definition 14 A complex-valued function on G is a *character* of the group G if it is central, nonnegative definite (as a function on the group algebra of G), and equal to 1 at the identity element of G.

Definition 15 The K_0-*functor* $\mathrm{K}_0(G)$ of the group G is the ordered Abelian group with order identity formed by all equivalence classes of finitely generated projective virtual complex modules over the group G; the cone of nonnegative elements of the K_0-functor is the cone of proper modules; the order identity of the K_0-functor is the module that corresponds to the regular representation of the group G.

Definition 16 A complex-valued homomorphism on $\mathrm{K}_0(G)$, i. e., a homomorphism of Abelian groups between $\mathrm{K}_0(G)$ and $\mathbb{C}$, is a *trace* on the K_0-functor of the group G if it is nonnegative and equal to 1 at the order identity of $\mathrm{K}_0(G)$ (more precisely, it is a *finite trace*, but we do not consider other traces).

From [6] it follows that the set of all characters of G and the set of all traces on $\mathrm{K}_0(G)$ are Choquet simplices in the topological vector space of complex-valued central functions on G and complex-valued homomorphisms on $\mathrm{K}_0(G)$, respectively. The Choquet boundaries of these simplices consist of their indecomposable points, i. e., of indecomposable characters and indecomposable traces (recall that a point of a Choquet simplex is called indecomposable if it cannot be decomposed into a nontrivial convex combination of other points of the simplex). Denote by $\mathrm{ex}\mathcal{X}(G)$ the set of indecomposable characters of G (this notation means the Choquet boundary of the set $\mathcal{X}(G)$ of all characters of G).

Further, as shown in [1], there exists an isomorphism of topological vector spaces between the mentioned spaces that also establishes a homeomorphism between the sets of indecomposable characters of G and indecomposable traces on $\mathrm{K}_0(G)$, so the problems of describing these sets are equivalent.

In the case where the group G is finite, the solution of these problems can be given in terms of the elementary representation theory. Namely, the indecomposable characters of a finite group are exactly all normalized (divided by the dimension) traces of its irreducible representations.

The general idea of the representation theory of inductive limits of finite groups is to approximate different objects related to a limit group by similar objects related to its finite subgroups.

For instance, as shown in [6], the group $\mathrm{K}_0(G)$ is the inductive limit (in the category of ordered Abelian groups with order identities) of the groups $\mathrm{K}_0(G_i)$, $i \in I$, with respect to natural embeddings that can be defined in terms of the operation of induction of representations.

For $G = \mathrm{R}$, this fact implies that $\mathrm{K}_0(\mathrm{R})$ is isomorphic to the quotient of the group $\bigoplus_{n\in\mathbb{N}} \mathrm{K}_0(\mathrm{S}_n)$ by its subgroup generated by the elements

$$\pi - \mathrm{ind}_{\mathrm{S}_l}^{\mathrm{S}_{lm}}(\pi) \quad \text{for all } l,m \in \mathbb{N} \text{ and } \pi \in \mathrm{K}_0(\mathrm{S}_l) \tag{2}$$

(here we assume that S_l is periodically embedded into S_{lm}). The set $\mathrm{ex}\mathcal{X}(\mathrm{R})$, which is most important for us, can also be described in terms of the finite sets $\mathrm{ex}\mathcal{X}(\mathrm{S}_n)$, $n \in \mathbb{N}$. This description follows from the ergodic method for computing indecomposable characters, which will be stated below.

§ 3. The structure of a Riesz ring on $K_0(R)$ and a series of indecomposable characters of the group R

In this section, we define the structure of a Riesz ring on $K_0(R)$. Using it, we formulate a criterion for indecomposability of traces on $K_0(R)$ and describe a series of indecomposable characters of R.

To define a structure of a Riesz ring on $K_0(R)$, we need to introduce on it a nonnegative multiplication such that the identity element with respect to this multiplication is the order identity of $K_0(R)$; for this we will use the following operation on representations of symmetric groups.

Definition 17 The *exterior product* $\pi \boxtimes \rho \in K_0(S_{lm})$ of representations $\pi \in K_0(S_l)$ and $\rho \in K_0(S_m)$, *which corresponds to the operation of the direct product* (1) *of permutations*, is the representation $\mathrm{ind}_{S_l \times S_m}^{S_{lm}}(\pi \otimes \rho)$ (here we assume that the group $S_l \times S_m$ is embedded into the group S_{lm} using the operation $\sqcap$).

It is easy to see that the extension by linearity of the operation $\boxtimes$ to the Abelian group $\bigoplus_{n\in\mathbb{N}} K_0(S_n)$ defines on it the structure of an associative, commutative, and unital ring. Further, as we already know, the group $K_0(R)$ is isomorphic to the quotient of the group $\bigoplus_{n\in\mathbb{N}} K_0(S_n)$ by its subgroup generated by the elements (2). From [1] it follows that this subgroup is an ideal of the above-mentioned ring. Hence the operation $\boxtimes$ determines an associative and commutative multiplication on the quotient of the ring $\bigoplus_{n\in\mathbb{N}} K_0(S_n)$ by this ideal, i. e., on $K_0(R)$. This multiplication defines the required structure of a Riesz ring on $K_0(R)$. Further, there is a general criterion for indecomposability of traces on a Riesz ring (see [7, Sec. 3.1]); below we formulate it in the case of $K_0(R)$.

Theorem 2 *Let τ be a trace on $K_0(R)$. Then τ is indecomposable if and only if τ is a complex-valued character of the ring $K_0(R)$, i. e., a homomorphism of unital rings between $K_0(R)$ and $\mathbb{C}$.*

Using the above-mentioned homeomorphism between the sets of indecomposable characters of R and indecomposable traces on $K_0(R)$, one can reformulate this criterion in terms of characters of the group R, see [1]. To do this, we also need the following bijection. Let χ be a character of R; define a function $\widetilde{\chi}$ on the set $\bigcup_{n\in\mathbb{N}} S_n$ as follows. For $u \in S_n$, set $\widetilde{\chi}(u) = \chi(g)$, where $g \in R$ acts on the segment by formula (1). Obviously, for all $l, m \in \mathbb{N}$ and $u \in S_l$ we have: $\widetilde{\chi}|_{S_l}$ is a character of S_l, and $\widetilde{\chi}(u) = \widetilde{\chi}(u \sqcap \mathrm{id}_{\underline{m}})$. Conversely, a function on the set $\bigcup_{n\in\mathbb{N}} S_n$ that has these properties uniquely determines a character of the group R.

Corollary 1 *Let χ be a character of R. Then χ is indecomposable if and only if $\widetilde{\chi}$ is a multiplicative function with respect to the operation $\sqcap$, i. e., $\widetilde{\chi}(u \sqcap v) = \widetilde{\chi}(u)\,\widetilde{\chi}(v)$ for all $l, m \in \mathbb{N}$, $u \in S_l$, $v \in S_m$.*

Now we can construct a series of characters of the group R and prove their indecomposability.

Definition 18 The *natural character* χ_{nat} of the group R is defined for any rational rearrangement $g \in \mathrm{R}$ by the formula $\chi_{\mathrm{nat}}(g) = \mu(\{x \in [0,1) \mid g(x) = x\})$.

Theorem 3 *The functions* χ_{nat}^k *are indecomposable characters of the group* R *for all* $k \in \mathbb{N} \cup \{0, \infty\}$.

Proof. Clearly, $\chi_{\mathrm{nat}}^0 = 1$ is the trivial character of R, and $\chi_{\mathrm{nat}}^\infty = \lim_{k\to\infty} \chi_{\mathrm{nat}}^k = \delta_{\mathrm{id}}$ is the character of its regular representation. Now, for any $k, n \in \mathbb{N}$, let us compute the value at $u \in \mathrm{S}_n$ of the function $\widetilde{\chi_{\mathrm{nat}}^k}$ corresponding to χ_{nat}^k (here and below $\mathrm{Fix}(u)$ means the fixed point set of u):

$$\begin{aligned}
\widetilde{\chi_{\mathrm{nat}}^k}(u) &= \mu\left(\left\{x \in [0,1) \,\middle|\, \frac{u(\lfloor xn \rfloor) + \{xn\}}{n} = x\right\}\right)^k \\
&= \mu\left(\left\{x \in [0,1) \,\middle|\, u(\lfloor xn \rfloor) = \lfloor xn \rfloor\right\}\right)^k \\
&= \left(\frac{1}{n}\,|\mathrm{Fix}(u)|\right)^k .
\end{aligned}$$

From the obtained formula we see that the restriction of the function $\widetilde{\chi_{\mathrm{nat}}^k}$ to the group S_n is the normalized character of the kth tensor power of the natural representation of the group S_n, hence it is a character of this group, and, therefore, the function χ_{nat}^k is a character of the group R.

It can easily be proved that $|\mathrm{Fix}(u \sqcap v)| = |\mathrm{Fix}(u)|\,|\mathrm{Fix}(v)|$ for all $l, m \in \mathbb{N}$, $u \in \mathrm{S}_l$, and $v \in \mathrm{S}_m$, thus the function $\widetilde{\chi_{\mathrm{nat}}^k}$ is multiplicative, and, therefore, the character χ_{nat}^k is indecomposable. □

Using the criterion for indecomposability of characters of the group R, one can obtain some general properties of indecomposable characters of this group, see [1]. Unfortunately, it seems too difficult to obtain their complete description in that way, so that below we will consider quite another approach.

§ 4. The ergodic method for computing indecomposable characters

In this section, we describe the ergodic method for computing indecomposable characters of an inductive limit of finite groups in the case where this limit is the group R.

Let us introduce the main notion of the ergodic method (also applied to the group R).

Definition 19 A net $(\chi_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{ex}\mathcal{X}(\mathrm{S}_n)$ of indecomposable characters of the symmetric groups is called *weakly convergent with respect to the periodic embeddings* if there exists a complex-valued function ψ on the set $\bigcup_{n\in\mathbb{N}} \mathrm{S}_n$ such that for all $l \in \mathbb{N}$ and $u \in \mathrm{S}_l$ the net $\big(\chi_{lm}(u \sqcap \mathrm{id}_{\underline{m}})\big)_{m\in\mathbb{N}}$ tends to $\psi(u)$; the function ψ is called the *limit* of $(\chi_n)_{n\in\mathbb{N}}$. Here $\mathbb{N}$ is ordered by the divisibility relation.

Clearly, if ψ is the limit of the net $(\chi_n)_{n\in\mathbb{N}}$, then $\psi(u) = \psi(u \sqcap \mathrm{id}_{\underline{m}})$ for all $l, m \in \mathbb{N}$ and $u \in \mathrm{S}_l$, so that the function ψ gives rise to a well-defined function on the group R. Further, we exactly formulate how the set $\mathrm{ex}\mathcal{X}(\mathrm{R})$ can be described in terms of the finite sets $\mathrm{ex}\mathcal{X}(\mathrm{S}_n)$, $n \in \mathbb{N}$.

Theorem 4 *Let a function on the group* R *be determined by the limit of a weakly convergent net of indecomposable characters of the symmetric groups. Then this function is an indecomposable character of the group* R. *Conversely, if* χ *is an indecomposable character of the group* R, *then there exists a weakly convergent net of indecomposable characters of the symmetric groups that tends to the function* $\widetilde{\chi}$ *determined by the function* χ.

Proof. This statement is a special case of a general theorem on approximation of indecomposable characters. This theorem is proved for analogous cases in [5, § 4] and [6, Chap. 1, § 9]. □

At the present time, our main problem is to prove, using the ergodic method, that all indecomposable characters of R are exactly powers of its natural character χ_{nat}; below we show that this conjecture is a corollary of the following estimation. To formulate it, recall that for all $n \in \mathbb{N}$ the set of irreducible representations of the group S_n is parameterized by the set $\mathbb{Y}_n$ of Young diagrams with n cells. If $\lambda \in \mathbb{Y}_n$, then denote by χ^λ the normalized character of the irreducible representation of S_n corresponding to λ. Denote by $\mathrm{r}_1(\lambda)$ and $\mathrm{c}_1(\lambda)$ the number of cells in the first row and column of λ, respectively.

Conjecture. For all $l \in \mathbb{N}$ and $u \in \mathrm{S}_l$ there exists a constant C depending only on u such that we have

$$\left| \chi^\lambda(u \sqcap \mathrm{id}_{\underline{m}}) - \left(\frac{1}{l}\, |\mathrm{Fix}(u)| \right)^{lm - \mathrm{r}_1(\lambda)} \right| \leqslant \frac{C}{m} \tag{3}$$

for all $m \in \mathbb{N}$ and $\lambda \in \mathbb{Y}_{lm}$ with $\mathrm{r}_1(\lambda) \geqslant \mathrm{c}_1(\lambda)$.

Corollary 2 *A net* $(\chi^{\lambda_n})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{ex}\mathcal{X}(\mathrm{S}_n)$ *of indecomposable characters of the symmetric groups is weakly convergent if and only if the net*

$$\big(n - \max\{\mathrm{r}_1(\lambda_n), \mathrm{c}_1(\lambda_n)\}\big)_{n\in\mathbb{N}}$$

has a limit $k \in \mathbb{N} \cup \{0, \infty\}$; *in the case where the limit of the net* $(\chi^{\lambda_n})_{n\in\mathbb{N}}$ *exists, it determines the character* χ^k_{nat} *on the group* R.

Proof. Using the operation of transposition of diagrams, it is easy to see that for all $l \in \mathbb{N}$ and $u \in \mathrm{A}_l$ the conjectured estimation (3) implies the similar estimation (3) where $\mathrm{r}_1(\lambda)$ has to be replaced by $\max\{\mathrm{r}_1(\lambda), \mathrm{c}_1(\lambda)\}$ for all $m \in \mathbb{N}$ and $\lambda \in \mathbb{Y}_{lm}$. Obviously, from this we see that the existence of $\lim_m \chi^{\lambda_{lm}}(u \sqcap \mathrm{id}_{\underline{m}})$ is equivalent to the existence of $\lim_m(lm - \max(\mathrm{r}_1(\lambda_{lm}), \mathrm{c}_1(\lambda_{lm}))$; finally, if $u \in \mathrm{S}_l \setminus \mathrm{A}_l$, then replace it by $u \sqcap \mathrm{id}_{\underline{2}} \in \mathrm{A}_{2l}$. $\square$

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