FINITE DIMENSIONAL ANALYSIS AND POLYNOMIAL QUANTIZATION ON A HYPERBOLOID OF ONE SHEET

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This paper has two goals. The first one is to study "finite dimensional analysis" on the hyperboloid $\mathcal{X}$ of one sheet in $\mathbb{R}^3$,i.e. the decomposition of representations of the group $G = \mathrm{SL}(2,\mathbb{R})$ (or its factor-group $\mathrm{SO}_0(1,2)$) acting on spaces $\mathcal{A}_{2l}$ of restrictions to $\mathcal{X}$ of polynomials of degree $\leqslant 2l$. These representations (equipped a G-invariant bilinear form) can be regarded as a finite dimensional analogue of the so-called canonical representations, see, for example, [4], [1].

We solve this problem with using a machinery of "usual", infinite dimensional, harmonic analysis (see, for example, [5]): H-invariants, Poisson and Fourier transforms, spherical functions, a Plancherel formula.

The decomposition in question appears also when we tensor a finite dimensional irreducible representation π_l of G by its contragradient $\hat{\pi}_l$. Although the formula for the decomposition of $\pi_l \otimes \hat{\pi}_l$ is well-known ($\pi_l \otimes \hat{\pi}_l = \pi_0 + \pi_1 + \ldots + \pi_{2l}$), explicit constructions and formulae given in this paper are interesting both themselves (for example, a differential formula for the Poisson transform, etc.) and as a ruling example for generalizations.

The second goal is to construct a polynomial quantization (symbol calculus) for the hyperboloid $\mathcal{X}$, i.e. quantization on the space $S(\mathcal{X})$ of polynomials on $\mathcal{X}$. For almost all values of a parameter $\sigma \in \mathbb{C}$ we introduce in $S(\mathcal{X})$ a structure of an associative algebra and establish that the corresponding principle is true (for "the Planck's constant" h one has to take $-1/2\sigma$). The space $S(\mathcal{X})$ is formed by covariant symbols of operators corresponding to elements of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G in a representation T_σ. We define also contravariant symbols for the same operators. It gives us two transforms: $\mathcal{O}$ and $\mathcal{B}$. The first one, the transform $\mathcal{O}$, is a map on a space of operators ($A = \mathcal{O}D$ means that operators D and A have one and the same polynomial F as the covariant and the contravariant symbol, respectively). The second one, the Berezin transform $\mathcal{B}$, is a transform of $S(\mathcal{X})$, it carries contravariant symbols to covariant symbols. We give full and explicit descriptions of these transforms $\mathcal{O}$ and $\mathcal{B}$. For example, for $\mathcal{B}$, we write the spectral decomposition, an explicit expresion in terms of the Laplacian, an explicit "deformation" decompositon.

Some fragments of this work were published in [7], [8].

The theory developed in the present paper can be generalized to para-Hermitian symmetric spaces in a very natural way. It will be published elsewhere.

§1. Representations of the group $\mathrm{SL}(2,\mathbb{R})$

In this Section we recall some material about representations of the group $G = \mathrm{SL}(2,\mathbb{R})$, see, for example, [3]. The group G consists of real 2×2 matrices

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \alpha\delta - \beta\gamma = 1. \tag{1.1}$$

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Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}$. Take the following basis of $\mathfrak{g}$:

$$L_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},\ L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.2}$$

The centre of $\mathfrak{U}$ is generated by the element

$$\Delta_{\mathfrak{g}} = L_1^2 + \frac{1}{2}(L_+L_- + L_-L_+)$$

For $\lambda \in \mathbb{C}, \varepsilon = 0, 1$, we denote

$$t^{\lambda,\varepsilon} = |t|^\lambda \mathrm{sgn}^\varepsilon t = t_+^\lambda + (-1)^\varepsilon t_-^\lambda,$$

a character of $\mathbb{R}^*$. We shall also use the following symbols for "generalized powers":

$$a^{(n)} = a(a-1)\ldots(a-n+1),\ a^{[n]} = a(a+1)\ldots(a+n-1). \tag{1.3}$$

For $\sigma \in \mathbb{C}, \varepsilon = 0, 1$, let $\mathcal{D}_{\sigma,\varepsilon}$ denote the space of complex valued C^∞ functions $f(x)$ on $\mathbb{R}$ such that the function

$$\hat{f}(t) = t^{2\sigma,\varepsilon} f(1/t) \tag{1.4}$$

is also of class C^∞. For the topology on $\mathcal{D}_{\sigma,\varepsilon}$, see, for example, [3]. The representation $T_{\sigma,\varepsilon}$ of the group G acts on $\mathcal{D}_{\sigma,\varepsilon}$ by

$$(T_{\sigma,\varepsilon}(g)f)(t) = f(\tilde{t})(\beta t + \delta)^{2\sigma,\varepsilon} \tag{1.5}$$

where

$$\tilde{t} = t \cdot g = \frac{\alpha t + \gamma}{\beta t + \delta} \tag{1.6}$$

Permuting in (1.5) α with δ and β with γ, we obtain the *contragradient* representation $\hat{T}_{\sigma,\varepsilon}$:

$$\hat{T}_{\sigma,\varepsilon}(g)f)(t) = f(\hat{t})(\gamma t + \alpha)^{2\sigma,\varepsilon}$$

where

$$\hat{t} = t \cdot \hat{g} = \frac{\delta t + \beta}{\gamma t + \alpha},\ \hat{g} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}, \tag{1.7}$$

The representations $T_{\sigma,\varepsilon}$ and $\hat{T}_{\sigma,\varepsilon}$ are equivalent, the equivalence is given by the map $f \mapsto \hat{f}$, see (1.4).

A differentiable representation of G gives rise to a representation of $\mathfrak{g}$ and a representation of $\mathfrak{U}$. We retain for them the symbol of the representation of G.

In particular, we have

$$T_{\sigma,\varepsilon}(L_1) = t\frac{d}{dt} - \sigma \tag{1.8}$$

$$T_{\sigma,\varepsilon}(L_+) = t^2\frac{d}{dt} + 2\sigma t \tag{1.9}$$

$$T_{\sigma,\varepsilon}(L_-) = \frac{d}{dt} \tag{1.10}$$

For the contragradient representation one has to multiply by -1 the right hand side of (1.8) and to permute the right hand sides of (1.9) and (1.10).

The element $\Delta_{\mathfrak{g}}$ goes to a scalar operator:

$$T_{\sigma,\varepsilon}(\Delta_{\mathfrak{g}}) = \hat{T}_{\sigma,\varepsilon}(\Delta_{\mathfrak{g}}) = \sigma(\sigma+1)E \tag{1.11}$$

The bilinear form

$$\langle F, f\rangle = \int_{-\infty}^{\infty} F(t)f(t)dt \tag{1.12}$$

is invariant with respect to the pair $(T_{\sigma,\varepsilon}, T_{-\sigma-1,\varepsilon})$, so that

$$\langle T_{\sigma,\varepsilon}(g)F, f\rangle = \langle F, T_{-\sigma-1,\varepsilon}(g^{-1})f\rangle \tag{1.13}$$

The operator with the kernel $(t-s)^{-2\sigma-2,\varepsilon}$ intertwines $T_{\sigma,\varepsilon}$ and $T_{-\sigma-1,\varepsilon}$ (for points σ where this operator has poles, one has to take the first Laurent coefficient). If $T_{\sigma,\varepsilon}$ is irreducible, then it is equivalent to $T_{-\sigma-1,\varepsilon}$. It will be more convenient to us to deal with other intertwining operator:

$$(A_{\sigma,\varepsilon}f)(t)=\int_{-\infty}^{\infty}(1-ts)^{-2\sigma-2,\varepsilon}f(s)\,ds$$

which intertwines $T_{\sigma,\varepsilon}$ and $\hat{T}_{-\sigma-1,\varepsilon}$:

$$\hat{T}_{-\sigma-1,\varepsilon}(g)A_{\sigma,\varepsilon}=A_{\sigma,\varepsilon}T_{\sigma,\varepsilon}(g),\; g\in G,$$

and also $\hat{T}_{\sigma,\varepsilon}$ and $T_{-\sigma-1,\varepsilon}$. The composition of these operators there and back is a scalar operator:

$$A_{-\sigma-1,\varepsilon}A_{\sigma,\varepsilon}=\omega_0(\sigma,\varepsilon)E \tag{1.14}$$

where

$$\omega_0(\sigma,\varepsilon)=\frac{2\pi}{2\sigma+1}\operatorname{tg}((2\sigma-\varepsilon)\pi/2)$$

The operator $A_{\sigma,\varepsilon}$ is self-dual with respect to the form (1.12):

$$\langle A_{\sigma,\varepsilon}F,f\rangle=\langle F,A_{\sigma,\varepsilon}f\rangle. \tag{1.15}$$

Let us extend the representation $T_{\sigma,\varepsilon}$ to the dual space $\mathcal{D}'_{\sigma,\varepsilon}$ (linear continuous functionals, distributions) by formula (1.13) where $F\in\mathcal{D}'_{\sigma,\varepsilon}$ and $f\in\mathcal{D}_{-\sigma-1,\varepsilon}$ ($\mathcal{D}_{\sigma,\varepsilon}$ is embedded in $\mathcal{D}'_{\sigma,\varepsilon}$ by assigning to $F\in\mathcal{D}_{\sigma,\varepsilon}$ the functional $f\mapsto\langle F,f\rangle$). Similarly we extend the operator $A_{\sigma,\varepsilon}$ – by means of (1.15).

Formulae (1.8) – (1.10) for the representation $T_{\sigma,\varepsilon}$ on $\mathcal{D}'_{\sigma,\varepsilon}$ generate a representation T_σ (notice that it does not depend on ε) of $\mathfrak{g}$ given by the same formulae. We shall assume that this representation acts on one of the spaces: C^∞ functions (defined on $\mathbb{R}$ or on intervals); the space P of polynomials; the space Z of distributions, concentrated at zero. The two latter spaces are Verma moduli.

The action of $\mathfrak{g}$ on P in the basis t^k is given by:

$$T_\sigma(L_1)t^k=(k-\sigma)t^k, \tag{1.16}$$

$$T_\sigma(L_+)t^k=(2\sigma-k)t^{k+1}, \tag{1.17}$$

$$T_\sigma(L_-)t^k=kt^{k-1}, \tag{1.18}$$

The space Z consists of linear combinations of the delta function $\delta(x)$ and its derivatives. We have

$$T_\sigma(L_1)\delta^{(k)}=(-\sigma-1-k)\delta^{(k)},$$

$$T_\sigma(L_+)\delta^{(k)}=-k(2\sigma+k+1)\delta^{(k-1)},$$

$$T_\sigma(L_-)\delta^{(k)}=\delta^{(k+1)}.$$

The operator $A_{\sigma,\varepsilon}$ carries the basis t^k in P to the basis $\delta^{(k)}$ in Z (with factors):

$$\int_{-\infty}^{\infty}(1-ts)^{-2\sigma-2,\varepsilon}s^k\,ds=\omega_k(\sigma,\varepsilon)\delta^{(k)}(t), \tag{1.19}$$

where

$$\omega_k(\sigma,\varepsilon)=(-1)^k\frac{2\pi}{(2\sigma+1)^{[k+1]}}\operatorname{tg}((2\sigma-\varepsilon)\pi/2)$$

(for $k=0$ formula (1.19) is equivalent to (1.14) and for $k>0$ is obtained by the differentiation). And, inversely, the operator $A_{-\sigma-1,\varepsilon}$ carries the basis $\delta^{(k)}$ to the basis t^k (with factors):

$$A_{-\sigma-1,\varepsilon}\delta^{(k)}=(2\sigma)^{(k)}t^k \tag{1.20}$$

The representations $T_{\sigma,\varepsilon}$ are irreducible except the case $2\sigma \in \mathbb{Z}, \varepsilon \equiv 2\sigma$ (here and further the sign $\equiv$ means the congruence modulo 2). Denote $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $2\sigma \in \mathbb{N}$, $\varepsilon \equiv 2\sigma$, the representation $T_{\sigma,\varepsilon}$ has an invariant finite dimensional subspace V_σ, consisting of polynomials in t of degree $\leqslant 2\sigma$, so that $\dim V_\sigma = 2\sigma + 1$. In this case let π_σ denote the restriction $T_{\sigma,\varepsilon}$ to V_σ. On the basis t^k of V_σ it acts by formulae (1.16) – (1.18) – with replacing T_σ by π_σ.

The operator $A_{\sigma,\varepsilon}$ vanishes on V_σ, and its derivative

$$A_\sigma' = \frac{\partial}{\partial\tau} A_{\tau,\varepsilon}\Big|_{\tau=\sigma,\varepsilon\equiv 2\sigma} \tag{1.21}$$

gives rise to an equivalence of V_σ and the finite dimensional quotient space for $T_{-\sigma-1,\varepsilon}$. Moreover, we have on V_σ:

$$A_{-\sigma-1,\varepsilon} A_\sigma' = -\frac{\pi^2}{2\sigma+1} E \tag{1.22}$$

Any irreducible finite dimensional representation of G (and of $\mathfrak{g}$) is equivalent to one of π_σ. We shall say that the basis $e_0, e_1, \ldots, e_{2\sigma}$ in the representation space is *standard*, if the action of L_1, L_+, L_- in this basis is the same as (1.16) – (1.18) in the basis $1, t, \ldots, t^{2\sigma}$ in V_σ. The standard basis is unique up to the factor. The vector e_0 is *minimal*: $L_- e_0 = 0$, and the vector $e_{2\sigma}$ is *maximal*: $L_+ e_{2\sigma} = 0$. A vector e_m is obtained from the minimal vector e_0 as follows:

$$e_m = \frac{1}{(2\sigma)^{(m)}} L_+^m e_0.$$

Let $2\sigma \in \mathbb{N}$. Then the module Z with respect to $T_{-\sigma-1}$ or $\hat{T}_{-\sigma-1}$ has a $\mathfrak{g}$-invariant submodule spanned by $\delta^{(k)}$ with $k \geqslant 2\sigma+1$. The operator $A_{-\sigma-1,\varepsilon}$ vanishes on it (see (1.20)) and maps the quotient module $Z_{-\sigma-1}$ generated by $\delta^{(k)}, 0 \leqslant k \leqslant 2\sigma$, onto V_σ. The operator A_σ' carries the basis t^k in V_σ to the standard basis $[(2\sigma)^{(k)}]^{-1}\delta^{(k)}$ in $Z_{-\sigma-1}$ with the factor (cf. (1.22)):

$$A_\sigma' t^k = -\frac{\pi^2}{2\sigma+1} \cdot \frac{1}{(2\sigma)^{(k)}} \delta^{(k)}(t)$$

For any irreducible subfactor of $T_{\sigma,\varepsilon}$ there exists a unique up to the factor G-invariant bilinear form. In particular, such a form B_σ on V_σ is defined by

$$B_\sigma(t^k, t^m) = (-1)^k \binom{2\sigma}{k}^{-1} \delta_{m,2\sigma-k} \tag{1.23}$$

Let us else write a bilinear form B_σ' on V_σ which is G-invariant with respect to the pair $(\pi_\sigma, \hat{\pi}_\sigma)$:

$$B_\sigma'(t^k, t^m) = (-1)^k \binom{2\sigma}{k}^{-1} \delta_{m,k} \tag{1.24}$$

This form can be written as follows:

$$B_\sigma(f_1, f_2) = -\frac{2\sigma+1}{\pi^2} \langle A_\sigma' f_1, f_2 \rangle \tag{1.25}$$

§2. Tensor product $\pi_l \otimes \hat{\pi}_l$

Let $2l \in \mathbb{N}$. The representation $R_l = \pi_l \otimes \hat{\pi}_l$ of G acts on the space $W_l = V_l \otimes V_l$ which consists of polynomials $f(\xi, \eta)$ in two variables ξ and η of degree $\leqslant 2l$ with respect to each of them by

$$R_l(g) f(\xi, \eta) = f(\tilde{\xi}, \hat{\eta}) \Big[(\beta\xi + \delta)(\gamma\eta + \alpha) \Big]^{2l},$$

see (1.6) and (1.7). The polynomial

$$N = 1 - \xi\eta \tag{2.1}$$

has the following property:

$$1 - \tilde{\xi}\hat{\eta} = \frac{1-\xi\eta}{(\beta\xi+\delta)(\gamma\eta+\alpha)}$$

Therefore, the polynomial

$$\Phi = N^{2l} = (1-\xi\eta)^{2l} \tag{2.2}$$

is fixed under R_l:

$$R_l(g)\Phi = \Phi \tag{2.3}$$

It is easy to check that for any $k = 0, 1, \ldots, 2l$ the polynomial

$$u_k = N^{2l-k}\eta^k, \tag{2.4}$$

is annihilated by L_- and is an eigenvector for L_1 with the eigenvalue $-k$. Therefore, it is a minimal vector and generates an irreducible subspace $W_l^{(k)}$ of W_l where π_k acts. Since the sum of dimensions is precisely the dimension of W_l: $1+3+5+\ldots+(4l+1) = (2l+1)^2$, we obtain the decomposition:

$$R_l = \pi_0 + \pi_1 + \ldots + \pi_{2l},$$

and, respectively,

$$W_l = W_l^{(0)} + W_l^{(1)} + \ldots + W_l^{(2l)}, \tag{2.5}$$

where

$$W_l^{(k)} = N^{2l-k} W_{k/2}^{(k)}$$

A standard basis in $W_l^{(k)}$ is $N^{2l}F_{k,m}$ where $F_{k,m}$ are functions defined by (4.4) below. In particular, the minimal and maximal vectors are u_k, see (2.4), and

$$v_k = N^{2l-k}\xi^k.$$

Take on W_l the bilinear invariant form $\mathcal{B}_l$ which is the "tensor square" of $\mathcal{B}_l'$, see (1.24), (1.25), namely, for pure tensors:

$$\mathcal{B}_l(\varphi\otimes\psi, \varphi_1\otimes\psi_1) = \mathcal{B}_l'(\varphi,\psi_1)\mathcal{B}_l'(\psi,\varphi_1)$$

so that

$$\mathcal{B}_l(\xi^k\eta^m, \xi^m\eta^k) = (-1)^{k+m}\binom{2l}{k}^{-1}\binom{2l}{m}^{-1}$$

and $\mathcal{B}_l$ is equal to zero for other pairs of basis elements $\xi^k\eta^m$. The form $\mathcal{B}_l$ is invariant with respect to R_l:

$$\mathcal{B}_l(R_l(g)f_1, R_l(g)f_2) = \mathcal{B}_l(f_1, f_2)$$

The subspaces $W_l^{(k)}$ are pairwise orthogonal with respect to $\mathcal{B}_l$.

Denote

$$\mu_{lk} = \mathcal{B}_l(u_k, v_k) \tag{2.6}$$

A computation with using the binomial formula for generalized powers (see, for example, [6], Sect. I, No. 35)

$$(a+b)^{[n]} = \sum_{j=0}^{n}\binom{n}{j}a^{[j]}b^{[n-j]} \tag{2.7}$$

gives

$$\mu_{lk} = (-1)^k\frac{k!^2(2l-k)!(2l+k+1)!}{(2k+1)!(2l)!^2} \tag{2.8}$$

§3. Hyperboloid of one sheet

In this section we show that the tensor product $R_l = \pi_l \otimes \hat{\pi}_l$ is equivalent to the representation of G by translations on a space of polynomials on the hyperboloid of one sheet in $\mathbb{R}^3$ (see Theorem 3.1).

Let us introduce in $\mathbb{R}^3$ the bilinear form

$$[x, y] = -x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{3.1}$$

Let $\mathcal{X}$ and $\mathcal{X}_0$ denote the hyperboloid $[x, x] = 1$ and the cone $[x, x] = 0, x \neq 0$, respectively. Let us realize $\mathbb{R}^3$ as the space of matrices

$$x = \frac{1}{2}\begin{pmatrix} 1 - x_3 & -x_1 + x_2 \\ x_1 + x_2 & 1 + x_3 \end{pmatrix}$$

Then $\det x = (1/4)(1 - [x, x])$. The group G acts on these matrices as follows:

$$x \mapsto g^{-1} x g \tag{3.2}$$

On $\mathcal{X}$, it acts transitively. The stabilizer of the basic point

$$x^0 = (0, 0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is the diagonal subgroup H of G. Under (3.2), where g is given by (1.1), the point x^0 goes to the point

$$x = (\alpha\gamma + \beta\delta,\ \alpha\gamma - \beta\delta,\ \alpha\delta + \beta\gamma)$$

The action (3.2) gives a right action of G on vectors $x \in \mathbb{R}^3$ by means of matrices from $SO_0(1, 2)$. So we get a homomorphism of G onto $SO_0(1, 2)$ (with the kernel $\pm E$).

The action of G on functions f on $\mathcal{X}$ by translations will be denoted by U:

$$U(g) f(x) = f(g^{-1} x g) \tag{3.3}$$

Introduce on $\mathcal{X}$ coordinates ξ, η:

$$x = N^{-1}(\xi + \eta, \xi - \eta, 1 + \xi\eta)$$

(for N, see (2.1)), so that in the matrix realization we have:

$$x = \frac{1}{N}\begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix}$$

These coordinates are defined on $\mathcal{X}$ except $x_3 = -1$. The action (3.2) is given by fractional linear transformations (separately in each variable ξ and η):

$$\xi \mapsto \tilde{\xi} = \xi \cdot g,\ \eta \mapsto \hat{\eta} = \eta \cdot \hat{g}$$

see (1.6) and (1.7), so that

$$U(g) f(\xi, \eta) = f(\tilde{\xi}, \hat{\eta})$$

The basic point x^0 has coordinates $\xi = 0, \eta = 0$. An element $g \in G$, see (1.1), carries x^0 to the point x with coordinates

$$\xi = \gamma/\delta,\ \eta = \beta/\alpha, \tag{3.4}$$

so that $N = 1/\alpha\delta$.

Write in ξ, η the G-invariant measure dx, the Laplace-Beltrami operator Δ and the Poisson bracket $\{\varphi, \psi\}$:

$$dx = dx(\xi, \eta) = N^{-2} d\xi d\eta \tag{3.5}$$

$$\Delta = U(\Delta_{\mathfrak{g}}) = N^2 \frac{\partial^2}{\partial\xi\partial\eta} \tag{3.6}$$

$$\{\varphi, \psi\} = N^2\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\psi}{\partial\eta} - \frac{\partial\varphi}{\partial\eta}\frac{\partial\psi}{\partial\xi}\right) \tag{3.7}$$

A polynomial f on $\mathbb{R}^3$ is called *harmonic* (with respect to (3.1)), if

$$\left(-\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) f = 0 \tag{3.8}$$

Let S and $\mathcal{H}$ denote the spaces of all polynomials and of harmonic polynomials, respectively. Let $\mathcal{H}_k$ denote the subspace of $\mathcal{H}$ consisting of homogeneous polynomials of degree k. We shall denote the restriction of $\mathcal{H}_k$ to $\mathcal{X}$ by $\mathcal{H}_k(\mathcal{X})$ – and similarly for other spaces. For harmonic polynomials this restriction is an one-to-one map. Besides it, we have $S(\mathcal{X}) = \mathcal{H}(\mathcal{X})$. The space $\mathcal{H}_k(\mathcal{X})$ is invariant and irreducible with respect to (3.3), the corresponding representation is equivalent to π_k. Equation (3.8) gives for polynomials from $\mathcal{H}_k(\mathcal{X})$ the condition (cf. (3.6) and (1.11)):

$$\Delta F = k(k+1)F \tag{3.9}$$

Denote

$$\mathcal{A}_k = \mathcal{H}_0(\mathcal{X}) + \mathcal{H}_1(\mathcal{X}) + \ldots \mathcal{H}_k(\mathcal{X}) \tag{3.10}$$

(the space of the restrictions to $\mathcal{X}$ of polynomials, or of harmonic polynomials, of degree $\leqslant k$).

Remember R_l from §2. Let us divide all polynomials $f(\xi,\eta)$ from W_l by Φ, see (2.2):

$$\mathcal{L}f = \Phi^{-1} f \tag{3.11}$$

We obtain the space $\Phi^{-1}W_l$ of some rational functions in ξ, η.

It follows from (2.3) that $\mathcal{L}$ intertwines R_l and the restriction of U to $\Phi^{-1}W_l$. In particular, (2.3) gives that the function $\mathbf{1}$ (identically equal to 1) is fixed under U.

Theorem 3.1. $\Phi^{-1}W_l = \mathcal{A}_{2l}$.

Proof. By virtue (3.10) and (2.5), it suffices to indicate in $\Phi^{-1}W_l$ at least one element from $\mathcal{H}_k(\mathcal{X})$ for each $k = 0, 1, \ldots, 2l$. Such an element is, for example, the minimal vector

$$\Phi^{-1}u_k = \left(\frac{\eta}{N}\right)^k = \left(\frac{x_1 - x_2}{2}\right)^k$$

(see (2.4)), because $(x_1 - x_2)^k$ belongs to $\mathcal{H}_k(\mathcal{X})$. □

§4. Finite dimensional harmonic analysis on hyperboloid

In this Section we give explicit constructions and formulae for the decomposition of the representation U of G on $\mathcal{A}_{2l}$ (and therefore of the tensor product $R_l = \pi_l \otimes \hat{\pi}_l$ from §2).

Let us transfer the bilinear form $\mathcal{B}_l$ from W_l to $\mathcal{A}_{2l}$ by means of $\mathcal{L}$, see (3.11), and retain the symbol, so that

$$\mathcal{B}_l(\Phi^{-1}\xi^k\eta^m, \Phi^{-1}\xi^m\eta^k) = (-1)^{k+m}\binom{2l}{k}^{-1}\binom{2l}{m}^{-1}$$

and $\mathcal{B}_l$ is equal to zero for other pairs of basis elements $\Phi^{-1}\xi^k\eta^m$, $k, m \leqslant 2l$.

Let us indicate intertwining operators between π_k and U on $\mathcal{A}_{2l}$ ($k \leqslant 2l$).

Recall that H is the diagonal subgroup of G. The subspace of H-invariants for π_k is non-trivial if and only if $k \in \mathbb{N}$. Then it is one-dimensional and is spanned by

$$\theta_k(t) = t^k.$$

The corresponding Poisson kernel is defined as follows:

$$P_k(x;t) = P_k(\xi,\eta;t) = (\pi_k(g^{-1})\theta_k)(t)$$

where $x = g^{-1}x^0g$ and ξ, η are obtained from g by (3.4). Here it is explicitly:

$$P_k(\xi,\eta;t) = \left[\frac{(t-\xi)(1-\eta t)}{N}\right]^k \tag{4.1}$$

or

$$P_k(x,t) = [x,y]^k,$$

where $y = ((t^2+1)/2, (t^2-1)/2), t)$, a point of the cone $\mathcal{X}_0$.

The Poisson transform $\mathcal{P}_k : V_k \to \mathcal{A}_{2l}$ (recall $k \leqslant 2l$) is defined by

$$(\mathcal{P}_k\varphi)(\xi,\eta) = B_k(\pi_k(g^{-1})\theta_k, \varphi) \tag{4.2}$$

where B_k is the bilinear form (1.23) on V_k. It is an intertwining operator:

$$\mathcal{P}_k\pi_k(g) = U(g)\mathcal{P}_k$$

So that, by (3.10), its image is $\mathcal{H}_k(\mathcal{X})$. It transfers the basis t^m in V_k to a standard basis in $\mathcal{H}_k(\mathcal{X})$. Let $F_{k,m}$ denote this standard basis multiplied by $(-1)^k$:

$$F_{k,m} = (-1)^k\mathcal{P}_k t^m \tag{4.3}$$

By (4.1), (4.2) and (1.23) we have

$$F_{k,m}(\xi,\eta) = N^{-k}\binom{2k}{m}^{-1}\sum_{j=0}^{k}\binom{k}{j}\binom{k}{m-j}\xi^{m-j}\eta^{k-j} \tag{4.4}$$

(in fact, the summation is taken over $0 \leqslant j \leqslant m$ for $m \leqslant k$ and over $m-k \leqslant j \leqslant k$ for $m \geqslant k$). In particular, the minimal and maximal vectors are

$$F_{k,0} = \left(\frac{\eta}{N}\right)^k = \left(\frac{x_1-x_2}{2}\right)^k, F_{k,2k} = \left(\frac{\xi}{N}\right)^k = \left(\frac{x_1+x_2}{2}\right)^k \tag{4.5}$$

Notice that

$$F_{k,m}(\xi,\eta) = F_{k,2m-k}(\eta,\xi)$$

Since $\mathcal{H}_k(\mathcal{X})$ is irreducible, values of $\mathcal{B}_l$ on the basis $F_{k,m}$ differ from values of B_k on the basis t^m by the factor only, which in virtue of (2.6) is equal to μ_{lk}, see (2.8), so that by (1.23) we have

$$\mathcal{B}_l(F_{k,m}, F_{k,2k-m}) = (-1)^m\binom{2k}{m}^{-1}\mu_{lk} \tag{4.6}$$

Returning to the Poisson kernel (4.1), we can now write it as follows:

$$P_k(\xi,\eta;t) = \sum_{m=0}^{2k}(-1)^{k+m}\binom{2k}{m}t^m F_{k,2k-m}(\xi,\eta) \tag{4.7}$$

so that for t fixed, it is an element of $\mathcal{H}_k(\mathcal{X})$.

Now define the Fourier transform $\mathcal{F}_k$: $\mathcal{A}_{2l} \to V_k$ as follows:

$$(\mathcal{F}_kF)(t) = \mathcal{B}_l(P_k(\cdot;t), F).$$

Introducing here (4.7), we obtain

$$(\mathcal{F}_kF)(t) = \sum_{m=0}^{2k}(-1)^{k+m}\binom{2k}{m}t^m\mathcal{B}_l(F_{k,2k-m}, F) \tag{4.8}$$

The Fourier transform $\mathcal{F}$ intertwines U and π_k:

$$\pi_k(g)\mathcal{F}_k = \mathcal{F}_k U(g),$$

and is conjugated to the Poisson transform:

$$\mathcal{B}_l(F, \mathcal{P}_k\varphi) = B_k(\mathcal{F}_k F, \varphi), \tag{4.9}$$

where $\varphi \in V_k$, $F \in \mathcal{A}_{2l}$; and the composition of these two transforms is a scalar operator:

$$\mathcal{F}_k\mathcal{P}_k = \mu_{lk}E$$

(these properties are obtained from (4.8) and properties of the Poisson transform).

Therefore, if $F \in \mathcal{H}_k(\mathcal{X})$, then

$$\mathcal{B}_l(F, F) = \mu_{l,k}^{-1} B_k(\mathcal{F}_k F, \mathcal{F}_k F)$$

and, for an arbitrary $F \in \mathcal{A}_{2l}$, we have

$$\mathcal{B}_l(F, F) = \sum_{k=0}^{2l} \mu_{l,k}^{-1} B_k(\mathcal{F}_k F, \mathcal{F}_k F).$$

It can be regarded as an analogue of a Plancherel formula, $\mu_{l,k}^{-1}$ being an analogue of a Plancherel measure.

The image of the H-invariant θ_k under the Poisson transform $\mathcal{P}_k$ is called the *spherical function*:

$$\Psi_k = \mathcal{P}_k\theta_k \tag{4.10}$$

By (4.3) we have $\Psi_k = (-1)^k F_{k,k}$ so that by (4.4)

$$\Psi_k = (-1)^k N^{-k} \binom{2k}{k}^{-1} \sum_{j=0}^{k} \binom{k}{j}^2 (\xi\eta)^{k-j}$$

$$= (-1)^k \binom{2k}{k}^{-1} P_k(x_3)$$

where P_k is the Legendre polynomial. The spherical function is H-invariant.

From (4.10) and (4.9) we have

$$\mathcal{B}(\Psi_k, F) = B_k(\theta_k, \mathcal{F}_k F)$$

A shifted spherical function $U(g^{-1})\Psi_k$ is an analogue of the Bergman kernel:

$$U(g^{-1})\Psi_k(u, v) = \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} F_{k,m}(u, v) F_{k,2k-m}(\xi, \eta) \tag{4.11}$$

where ξ, η correspond to g by (3.4).

For any H-invariant function $Q \in \mathcal{A}_{2l}$ we can define an operator in $\mathcal{A}_{2l}$ – the convolution $F \to Q \star F$ with Q:

$$(Q \star F)(\xi, \eta) = \mathcal{B}_l(U(g^{-1})Q, F)$$

where ξ, η correspond to g by (3.4).

In particular, the convolution with Φ^{-1} is the identity operator, so that Φ^{-1} plays the role of the delta function. The shifted function $U(g^{-1})\Phi^{-1}$ is the following function of two pairs variables (the Berezin kernel):

$$K_l(\xi, \eta; u, v) = \left[\frac{(1 - u\eta)(1 - \xi v)}{(1 - \xi\eta)(1 - uv)}\right]^{2l}$$

or, in terms of points of $\mathcal{X}$,

$$K_l(x, y) = \left(\frac{[x, y] + 1}{2}\right)^{2l}$$

Thus, this kernel K_l has the following reproducing property:

$$\mathcal{B}_l(K_l(x,\cdot), F(\cdot)) = F(x),\ F \in \mathcal{A}_{2l}$$

For the spherical function Ψ_k, the convolution with $\mu_{lk}^{-1}\Psi_k$ is the projection in $\mathcal{A}_{2l}$ onto $\mathcal{H}_k(\mathcal{X})$ (it follows from (4.11) and (4.6)).

Therefore, we have the decomposition:

$$\Phi^{-1} = \sum_{k=0}^{2l} \mu_{l,k}^{-1}\Psi_k \tag{4.12}$$

(it could be obtained also by a direct computation). Formula (4.12) also can be regarded as a Plancherel formula.

The same Poisson transform $\mathcal{P}_k$ can be obtained by using the representation T_{-k-1} on distributions (recall that its factor-representation on Z_{-k-1} is equivalent to π_k, see §1), then the Poisson transform is written in a differential form. Namely, H-invariant is $\delta^{(k)}(t)$, we take the bilinear form (1.12), so that:

$$(\mathcal{P}_k\varphi)(\xi,\eta) = c_k \int_{-\infty}^{\infty} T_{-k-1}(g^{-1})\delta^{(k)}(t)\varphi(t)dt$$

$$= c_k \int_{-\infty}^{\infty} \delta^{(k)}(t)T_k(g)\varphi(t)dt$$

$$= c_k(-1)^k \left(\frac{d}{dt}\right)^k\Big|_{t=0} \varphi\left(\frac{\alpha t+\gamma}{\beta t+\delta}\right)(\beta t+\delta)^{2k}.$$

where $c_k = [(2k)^{(k)}]^{-1} = k!/(2k)!$ and ξ,η are connected with g by (3.4). In particular, for basis function t^m we have

$$(\mathcal{P}_k t^m)(\xi,\eta) = c_k(-1)^k \left(\frac{d}{dt}\right)^k\Big|_{t=0} (\alpha t+\gamma)^m(\beta t+\delta)^{2k-m}$$

§5. Polynomial quantization on the hyperboloid

Let us apply to the hyperboloid $\mathcal{X}$ the scheme of quantization from [4], [1]. For an algebra of operators we take the algebra of operators $T_\sigma(X)$, $X \in \mathfrak{U}$, acting on functions of ξ, see §1. For a supercomplete systeme we take the kernel of the intertwining operator $A_{-\sigma-1,\varepsilon}$, namely,

$$\Phi(\xi,\eta) = \Phi_{\sigma,\varepsilon}(\xi,\eta) = N^{2\sigma,\varepsilon}$$

Let us call the function F of ξ,η, defined by

$$F(\xi,\eta) = \frac{1}{\Phi(\xi,\eta)} T_\sigma(X)\Phi(\xi,\eta), \tag{5.1}$$

the *covariant symbol* of the operator $T_\sigma(X)$. It does not depend on ε.

The covariant symbol of the identity operator is **1**, see §3. The covariant symbols of operators corresponding to the basis (1.2) of $\mathfrak{g}$ are $-\sigma x_3$, $\sigma(x_1+x_2)$, $-\sigma(x_1-x_2)$, respectively. The covariant symbol of the operator $T_\sigma(L_-^r) = (\partial/\partial\xi)^r$, see (1.13), is the minimal vector $F_{r,0}$ in $\mathcal{H}_r(\mathcal{X})$ up to the factor, namely,

$$\text{cov. symb. } T_\sigma(L_-^r) = (-1)^r(2\sigma)^r F_{r,0} \tag{5.2}$$

Lemma 5.1. *For any $X \in \mathfrak{U}$ of degree k the covariant symbol of the operator $T_\sigma(X)$ is a polynomial in $\mathcal{A}_k$ with coefficients depending on σ polynomially.*

The lemma follows from (1.8) – (1.10) immediately.

The operator $D(=T_\sigma(X))$ is reconstructed by its covariant symbol F in the following way (cf. [4], [1]). First we rewrite formula (1.14) as follows:

$$\varphi(\xi) = c \int \frac{\Phi(\xi, v)}{\Phi(u, v)} \varphi(u) dx(u, v)$$

where $c = 1/\omega_0(\sigma, \varepsilon)$ and $dx(u, v)$ is given by (3.5). Here and further integrals are taken over $\mathbb{R}^2$ or $\mathbb{R}$. Then by (5.1)

$$(D\varphi)(\xi) = c \int F(\xi, v) \frac{\Phi(\xi, v)}{\Phi(u, v)} \varphi(u) dx(u, v) \tag{5.3}$$

so that the kernel $K(\xi, u)$ of D is

$$K(\xi, u) = c \int F(\xi, v) \frac{\Phi(\xi, v)}{\Phi(u, v)} \frac{dv}{(1 - uv)^2}$$

Inversely, the covariant symbol is expressed by means of the kernel (we use (1.19) with $k = 0$):

$$F(\xi, \eta) = \frac{1}{\Phi(\xi, \eta)} \int K(\xi, u) \Phi(u, \eta) du$$

The correspondence $D \to F$ commutes with $\mathfrak{g}$: if F is the covariant symbol of the operator $D = T_\sigma(X)$, $X \in \mathfrak{U}$, then $U(L)F$, where $L \in \mathfrak{g}$, is the covariant symbol of the operator

$$T_\sigma(\mathrm{ad}L \cdot X) = [T_\sigma(L), D]$$

Theorem 5.2. *The set of the covariant symbols of all operators $T_\sigma(X), X \in \mathfrak{U}$, is the space $S(\mathcal{X})$, if $2\sigma \notin \mathbb{N}$, and the space $\mathcal{A}_{2\sigma}$, if $2\sigma \in \mathbb{N}$.*

The theorem follows from (5.2) and the $\mathfrak{g}$-equivariance of $D \to F$.

The multiplication of operators gives rise to a multiplication $*$ of covariant symbols: if $D = D_1 D_2$, then $F = F_1 * F_2$. This multiplication $*$ can be written as follows:

$$F_1 * F_2 = \frac{1}{\Phi} D_1(\Phi F_2) \tag{5.4}$$

(it includes (5.1) as a particular case), or, using (5.3) for (5.4),

$$(F_1 * F_2)(\xi, \eta) = \int F_1(\xi, v) F_2(u, \eta) \mathcal{B}(\xi, \eta; u, v) dx(u, v) \tag{5.5}$$

where

$$\mathcal{B}(\xi, \eta; u, v) = c \frac{\Phi(\xi, v)\Phi(u, \eta)}{\Phi(\xi, \eta)\Phi(u, v)},$$

the Berezin kernel. In terms of points $x \in \mathcal{X}$, the Berezin kernel is

$$\mathcal{B}(x; y) = c \left(\frac{[x, y] + 1}{2} \right)^{2\sigma, \varepsilon}$$

It is invariant with respect to translations:

$$\mathcal{B}(g^{-1}xg, g^{-1}yg) = \mathcal{B}(x, y) \tag{5.6}$$

Thus, we have

Theorem 5.3. *The set of covariant symbols indicated in Theorem 5.2 ($S(\mathcal{X})$ for $2\sigma \notin \mathbb{N}$, and $\mathcal{A}_{2\sigma}$ for $2\sigma \in \mathbb{N}$) is an associative algebra with the multiplication $*$.*

A monomial

$$\frac{\xi^k \eta^m}{N^r}, \quad k, m \leqslant r, \tag{5.7}$$

is the covariant symbol for the operator

$$D = \sum_{j=0}^{r-m} \binom{r-m}{j} (-1)^{m+j} \frac{1}{(2\sigma)^{(m+j)}} \xi^{k+j} \left(\frac{\partial}{\partial \xi}\right)^{m+j} = \tag{5.8}$$

$$= (-1)^r \frac{1}{(2\sigma)^{(r)}} \xi^k \left(\frac{\partial}{\partial \xi}\right)^m (\xi \frac{\partial}{\partial \xi} - 2\sigma)^{[r-m]}.$$

(it can be checked by applying of D to Φ). The kernel $K(\xi, u)$ of this operator is

$$K(\xi, u) = \sum_{j=0}^{r-m} \binom{r-m}{j} \frac{1}{(2\sigma)^{(m+j)}} \xi^{k+j} \delta^{(m+j)}(\xi - u)$$

Now define *contravariant symbols.* In accordance with a general scheme (see, for example, [1]), a function $F(\xi, \eta)$ is the contravariant symbol for the following operator A acting on functions $\varphi(\xi)$:

$$(A\varphi)(\xi) = c \int F(u, v) \frac{\Phi(\xi, v)}{\Phi(u, v)} \varphi(u) dx(u, v) \tag{5.9}$$

(notice the difference from (5.3) in the one argument only), so that the kernel $L(\xi, u)$ of the operator A is:

$$L(\xi, u) = c \int F(u, v) \frac{\Phi(\xi, v)}{\Phi(u, v)} \frac{dv}{(1 - uv)^2}$$

Inversely,

$$F(\xi, \eta) = \frac{1}{\Phi^*(\xi, \eta)} \int L(t, \xi) \Phi^*(t, \eta) dt$$

where we denote

$$\Phi^*(\xi, \eta) = \Phi_{-\sigma-1,\varepsilon}(\xi, \eta) = \frac{1}{\Phi(\xi, \eta)(1 - \xi\eta)^2}$$

For polynomials $F(\xi, \eta)$ from $S(\mathcal{X})$ corresponding operators A are differential operators. In particular, the monomial (5.7) is the contravariant symbol for the operator

$$A = \sum_{j=0}^{r-m} \binom{r-m}{j} \frac{1}{(-2\sigma - 2)^{(m+j)}} \left(\frac{\partial}{\partial \xi}\right)^{m+j} \circ \xi^{k+j} \tag{5.10}$$

(it is proved by a direct computation with using (1.19)), so that the kernel of this operator is

$$L(\xi, u) = \sum_{j=0}^{r-m} \binom{r-m}{j} \frac{1}{(-2\sigma - 2)^{(m+j)}} u^{k+j} \delta^{m+j}(u - \xi)$$

Thus we have the following diagram

$$\begin{array}{ccccccccc} \longrightarrow & D & \xrightarrow{\ \mathcal{O}\ } & A & \longrightarrow & & \\ \nearrow & & \searrow \quad \nearrow & & \searrow & & \nearrow \\ & \longrightarrow & F & \xrightarrow[\mathcal{B}]{} & F_1 & \longrightarrow & \end{array}$$

where arrows $\searrow$ are *co* and arrows $\nearrow$ are *contra*. Let us consider their compositions: $\mathcal{O} = contra \circ co$, $\mathcal{B} = co \circ contra$ (the latter is called the Berezin transform).

Theorem 5.4. *Let $A = \mathcal{O}(D)$ (i.e. there exists a polynomial $F \in S(\mathcal{X})$ which is the covariant and contravariant symbol for the operators D and A respectively). Then A is the transpose of D with respect to $d\xi$ with replacing σ by $-\sigma-1$:*

$$A = D^*\big|_{\sigma\to-\sigma-1} \tag{5.11}$$

To prove, one has to compare (5.8) and (5.10). On the language of kernels, (5.11) means the permutation of the arguments and the substitution $\sigma \to -\sigma - 1$:

$$L(\xi, u) = K(u,\xi)\big|_{\sigma\to-\sigma-1}$$

For $\mathcal{B}$ we have two theorems. The first one follows from (5.9) and (5.3).

Theorem 5.5. *Let $F_1 = \mathcal{B}F$ (i.e. F and F_1 from $S(\mathcal{X})$ are respectively the contravariant and covariant symbols of an operator A). Then F_1 is obtained from F by means of an integral operator with the Berezin kernel:*

$$F_1(\xi,\eta) = \int F(u,v)\mathcal{B}(\xi,\eta;u,v)dx(u,v)$$

In fact, for polynomials F the Berezin transform is reduced to a differential operator.

Theorem 5.6. *Let $F_1 = \mathcal{B}F$ (as in Theorem 5.5). Then F_1 is obtained from F by the following differential operator*

$$\mathcal{B}(\Delta) = \frac{\Gamma(-2\sigma+\tau)\Gamma(-2\sigma-\tau-1)}{\Gamma(-2\sigma)\Gamma(-2\sigma-1)}\bigg|_{\tau(\tau+1)=\Delta}$$

Proof. It follows from Theorem 5.5 and the invariance of $\mathcal{B}$, see (5.6), that the co- and contra-correspondences commute with $\mathfrak{g}$.

Therefore, it is sufficient to consider some polynomial F from $\mathcal{H}_k(\mathcal{X})$. For example, take the minimal one: $F = F_{k,0}$, see (4.5). Then, by (5.10), the operator A is

$$A = \frac{1}{(-2\sigma-2)^{(k)}}\left(\frac{d}{d\xi}\right)^k.$$

In its turn, A has, by (5.1), the covariant symbol

$$F_1 = b_k(\sigma)F \tag{5.12}$$

where

$$b_k(\sigma) = \frac{1}{(-2\sigma-2)^{(k)}}\cdot(2\sigma)^{(k)}(-1)^k \tag{5.13}$$

$$= \frac{\Gamma(-2\sigma+k)\Gamma(-2\sigma-k-1)}{\Gamma(-2\sigma)\Gamma(-2\sigma-1)} \tag{5.14}$$

Due to (3.9), equality (5.12) with (5.14) means $F_1 = \mathcal{B}(\Delta)F$ □

Theorem 5.6 is a spectral decomposition of the Berezin transform: on every $\mathcal{H}_k(\mathcal{X})$ it is the multiplication by $b_k(\sigma)$.

Let $\sigma \to -\infty$. Then by [2] 1.18(4), we have

$$\mathcal{B} \sim 1 - \frac{1}{2\sigma}\Delta$$

Therefore, by (5.5) and (3.6) we obtain

$$F_1 * F_2 = F_1F_2 - \frac{1}{2\sigma}N^2\frac{\partial F_1}{\partial\xi}\frac{\partial F_2}{\partial\eta} + \ldots$$

It gives us that on the algebra $S(\mathcal{X})$ the correspondence principle is true: if $\sigma \to -\infty$, then

$$F_1 * F_2 \to F_1F_2,$$

$$-2\sigma(F_1*F_2 - F_2*F_1) \to \{F_1, F_2\},$$

where $\{F_1, F_2\}$ is the Poisson bracket, see (3.7). So, for the Planck constant we take $h = -1/2\sigma$.

Moreover, we can write explicitly an asymptotic decomposition of $\mathcal{B}$ when $\sigma \to -\infty$. It is much more convenient to use not powers of h but "generalized powers", see (1.3), of $-2\sigma-2$.

Then the decomposition turns out to be given by a series which terminates on each $\mathcal{H}_k(\mathcal{X})$.

Theorem 5.7. *There is the following decomposition of the Berezin transform:*

$$\mathcal{B} = \sum_{m=0}^{\infty} \frac{\Delta(\Delta - 1\cdot 2)(\Delta - 2\cdot 3)\ldots(\Delta - (m-1)\cdot m)}{m!} \cdot \frac{1}{(-2\sigma - 2)^{(m)}} \tag{5.15}$$

Proof. The eigenvalue (5.13) of $\mathcal{B}$ can be written as

$$b_k = \frac{(\mu+2)^{[k]}}{\mu^{(k)}}$$

where $\mu = -2\sigma - 2$. By (2.7) we have

$$(\mu+2)^{[k]} = ((\mu - k + 1) + (k+1))^{[k]} = \sum_{m=o}^{k} \binom{k}{m} (\mu - k + 1)^{[k-m]} (k+1)^{[m]}$$

so that

$$b_k = \sum_{m=o}^{k} \binom{k}{m} (k+1)^{[m]} \frac{1}{\mu^{(m)}}$$

$$= \sum_{m=o}^{k} \frac{1}{m!} \frac{(k+m)!}{(k-m)!} \frac{1}{\mu^{(m)}}$$

The fraction $(k+m)!/(k-m)!$ can be written as $\prod_{j=1}^{m}(k^{[2]} - j^{(2)})$. But it is just the eigenvalue on $\mathcal{H}_k(\mathcal{X})$ of the nominator of the first fraction in (5.15). □

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