Invariant algebras of functions on spheres.

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Let $X = G/H$ be a compact homogeneous space; an invariant algebra on X is a closed subalgebra of $C(X)$ which is invariant with respect to the action of G on X. For example, the restiction to the skeleton (Shilov boundary) X of the algebra of analytic and continuous up to the boundary functions in a symmetric domain is an invariant algebra on X. There are many examples of this kind, and the question is if any nontrivial invariant algebra may be realized as an algebra of analytic (somewhere) functions.

The case of bi-invariant algebras on compact Lie groups is the most investigated one. Algebraic and topological properties of a group restricts the structure of invariant algebras on it. For example, Rider [15] proved that if a compact Lie group G admits an invariant Dirichlet algebra (this means that real parts of functions in A are dense in the space of all real-vaued continuous functions) then G is abelian and connected. Wolf [18] and Gangolli [2] showed that every uniformly closed bi-invariant algebra on a semisimple group is self-adjoint (a function algebra A is called self-adjoint if $\bar{A} = A$, where the bar denotes the complex conjugation; the algebra A is called antisymmetric if $A \cap \bar{A}$ contains only constant functions). Gichev [4] proved that a bi-invariant algebra A on a group G is antisymmetric if and only if the Haar measure of G is a multiplicative functional on A; Rosenberg [16] gave a characterization of bi-invariant antisymmetric algebras on compact groups in terms of harmonic analysis.

The case of invariant algebras on general homogeneous spaces G/H is not well-understood even for compact G. The hypothesis is that, for every nontrivial G-invariant algebra on G/H, there is a G-invariant foliation with leaves of the type $\tilde{G}/\tilde{H}$ such that the restriction of A to each leaf is some algebra of boundary values of holomorphic functions on a domain in $\tilde{G}^{\mathbb{C}}/\tilde{H}^{\mathbb{C}}$ whose skeleton is $\tilde{G}/\tilde{H}$. Such domains appears in a paper of Gel'fand and Gindikin [3]. They considered a real semisimple Lie group G as a boundary of a certain complex domain in $G^{\mathbb{C}}$. Ol'shanskiĭ [13] proved that these complex domains are interiors of subsemigroups of $G^{\mathbb{C}}$ of the form $G\exp(iC)$, where C is any $\mathrm{Ad}(G)$-invariant cone in the Lie algebra of G.

In this article we consider invariant algebras on spheres S^n; they are the simplest examples of compact homogeneous spaces. Montgomery, Samelson and Borel (see [14]) found all realizations of S^n as a homogeneous space G/H, where G is a compact group, H is its closed subgroup (the isotropy group of some point). Their results are summarized in the following table:

G	H	G/H
(1) $\mathrm{SO}(n+1)$	$\mathrm{SO}(n)$	S^n
(2) $\mathrm{U}(n)$	$\mathrm{U}(n-1)$	S^{2n-1}
(3) $\mathrm{SU}(n+1)$	$\mathrm{SU}(n)$	S^{2n+1}
(4) $\mathrm{Sp}(n)$	$\mathrm{Sp}(n-1)$	S^{4n-1}
(5) $\mathrm{Sp}(n) \times \mathrm{U}(1)$	$\mathrm{Sp}(n-1) \times \mathrm{U}(1)$	S^{4n-1}
(6) $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$	S^{4n-1}
(7) Spin_9	Spin_7	S^{15}
(8) Spin_7	G_2	S^7
(9) G_2	$\mathrm{SU}(3)$	S^6

Invariant algebras on $S^1 = \mathrm{SO}(2) = \mathrm{U}(1)$ are in one-to-one correspondence with subsemigroups of the group $\mathbb{Z}$ of integers [17]. K. de Leeuw and H. Mirkil [11] showed that there are only three $\mathrm{SO}(n+1)$-invariant algebras on S^n $(n > 1)$: the algebra $\mathbb{C}$ of constant functions, the algebra of even functions and $C(S^n)$. Note that they are self-adjoint. W. Rudin and A. Nagel characterized $\mathrm{U}(n)$-invariant algebras on S^{2n-1} ([17], [12]); J. Kane [9] described their maximal ideal spaces and realized almost all antisymmetric

U(n)-invariant algebras as algebras of holomorphic functions. We study invariant algebras on spheres in the remaining cases. Main results are stated in Theorems 1–3.

Theorem 1. *All invariant algebras in the cases (6)–(9) are self-adjoint. If $n > 2$ then* SU(n)*-invariant algebras on S^{2n-1} are* U(n)*-invariant. Every* Sp(n) $\times$ U(1)*-invariant antisymmetric algebra on S^{4n-1} is a subalgebra of some* U($2n$)*-invariant antisymmetric algebra.*

This theorem is proved in Propositions 2–7 (Propositions 2–4 are proved in Section 1, Proposition 5 — in Section 2, and Propositions 6–7 — in Section 3).

As a corollary of Propositions 5–7 we have a classification of invariant algebras on $P^n\mathbb{C}$ and $P^n\mathbb{H}$ (groups acting transitively on projective spaces were classified by Onishchik [14]). All these algebras are self-adjoint.

The case (4) is most complicated because the decomposition of the quasi-regular representation contains irreducible representations with a multiplicity > 1 ([10]). So in Section 4 we consider only the case $n = 1$ and only a family A_α of antisymmetric Sp(1)-invariant algebras of even functions on the three-dimensional sphere depending on a continuous parameter. These algebras are characterized by

Theorem 2. *There are invariant* CR*-conditions on S^3 such that A_α consists of all* CR*-functions. For every nonstandard invariant* CR*-conditions on S^3 there exists $\alpha > 0$ such that A_α is isomorphic to the algebra of all* CR*-functions; in particular, each* CR*-function f is even.*

These algebras differ from the invariant algebras on $\mathrm{U}(n)/\mathrm{U}(n-1)$. Every antisymmetric containing constant functions invariant algebra A on $\mathrm{U}(n)/\mathrm{U}(n-1)$ has the following properties:

a) A admits an invariant $\mathbb{Z}_+$-grading, i. e. $A = \oplus \sum_{k \in Z_+} A_k$, such that $A_i A_j \subseteq A_{i+j}$ where $\mathbb{Z}_+$ is the set of non-negative integers and A_i is an invariant (non-trivial) subspace for each i;

b) a linear functional corresponding to the normalized invariant measure is a multiplicative functional on A;

c) the group U(n) have a fixed point in the maximal ideal space of A.

An invariant $\mathbb{Z}_+$-grading is given by numbers $\{p - q : H(p,q) \subset A\}$. The properties b) and c) follow from [5]–[7]: every invariant algebra on $\mathrm{U}(n)/\mathrm{U}(n-1)$ is an averaging of a bi-invariant algebra on U(n). By the Theorem 1 every antisymmetric containing constant functions invariant algebra A on $\mathrm{Sp}(n) \times \mathrm{U}(1)/\mathrm{Sp}(n-1) \times \mathrm{U}(1)$ also has these properties.

Theorem 3. *A_α is an antisymmetric algebra isomorphic to the algebra of all analytic in the relative interior and continuous up to the boundary functions on the set*

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : 2|z_1|^2 + |z_2|^2 + 2|z_3|^2 \leq 1 + 2\alpha^2, z_2^2 - 4z_1 z_3 = 1\}$$

which coincides with its maximal ideal space M_α. For A_α each of the properties a), b), *and* c) *doesn't occur.*

In this paper invariant algebras on the spheres are studied by the following way. Since G is compact every invariant subspace X of $C(S^n)$ is uniquely determined by minimal ones included to X. They are finite dimensional complex linear spaces of polynomials. The action of G extends to the action of the complexification $G^{\mathbb{C}}$ in these spaces (representations of G, $G^{\mathbb{C}}$ and tangent representations of the corresponding Lie algebras will be denoted by the same letter). We find minimal invariant spaces and the corresponding highest weights of $g^{\mathbb{C}}$. Since any group G in the table above is naturally embedded to SO(n) or U(n) and the problem for these two groups is solved it is sufficient to find the decompositions of minimal SO(n)- or U(n)-invariant spaces. We find the highest vectors of the irreducible representations of G, i. e. the polynomials which are annihilated by n_+ ($g^{\mathbb{C}} = t \oplus n_+ \oplus n_-$, n_+ is the nilpotent subalgebra of g corresponding to positive roots, t is the Cartan subalgebra). To prove coincidence of minimal invariant spaces with SO(n)-invariant spaces we use the Weyl formula for the dimension d_λ of the irreducible representation of the Lie algebra g with the highest weight λ:

$$d_\lambda = \prod_{\beta > 0} \left(\frac{< \lambda, \beta >}{< \delta, \beta >} + 1 \right) \tag{1}$$

where δ is the half-sum of positive roots of g. Finally, we have to describe invariant subspaces which are closed under the multiplication; the complete solution to this problem is given in the cases (6)-(9), in the case (3) for $n > 1$ and a partial one in the case (5). In the cases (5)–(7) we use the Peter–Weyl theorem, the usual scalar product will be denoted by $\langle\ ,\ \rangle$.

We finish this introductionary part with the following remark. The problem of the description of all self-adjoint algebras has a geometrical interpretation.

Proposition 1. *A function algebra A is a self-adjoint G-invariant algebra on a homogeneous space $M = G/H$, G is compact, H is its closed subgroup, if and only if there exists a homogeneous space M' and a continuous equivariant mapping $\pi : M \to M'$ such that $A = C(M') \circ \pi$.*

Proof. Set $x \sim y$ if $f(x) = f(y)$ for all $f \in A$. Clearly, this is a G-invariant equivalence and A separates its classes. Hence $M' = M/\!\sim$ is a homogeneous space of G and A may be identified with $C(M')$ by the Stone–Weierstrass theorem.

Corollary. *Self-adjoint algebras on M are in one-to-one correspondence with closed subgroups of G which include H.*

We don't consider the geometrical problem of a description of all closed subgroups of a compact Lie group G which include a closed subgroup H but we receive a solution of this problem in the cases (5)–(9) as a consequence of the description of spectrums of self-adjoint invariant algebras.

1 Exceptional spheres

Let O_k be the space of homogeneous polynomials of degree k on S^{n-1} and H_k be its subspace of harmonic polynomials. The dimension of the space O_k is equal to $(n-1+k)!/(n-1)!k!$. Since $O_k = O_{k-2} \oplus H_k$ ([17]), the dimension of the space H_k is $(n-2+2k)!/(n-2)!k!$. Let ρ be the representation of $\mathrm{GL}(n, \mathbb{R})$ in O_k, $\rho(g)p(x) = p(xg)$. The tangent representation of $\mathrm{gl}(n, \mathbb{R})$ is defined by $\rho(e_{ij}) = x_i\partial/\partial x_j$ where e_{ij} is a matrix which element (i,j) is 1 and other elements are 0. The restriction of this representation to $\mathrm{SO}(n)$ is irreducible in H_k. The spaces H_k are invariant because $\mathrm{SO}(n)$ commutes with the Laplacian. The Lie algebra $\mathrm{so}(n, \mathbb{C})$ consists of all skew-symmetric matrices.

It is convenient to realize $\mathrm{so}(2m+1, \mathbb{C})$ as the set of matrices of the form $\begin{pmatrix} 0 & -U^t & -V^t \\ V & X & Y \\ U & Z & -X^t \end{pmatrix}$, where X is arbitrary, Y and Z are skew-symmetric $m \times m$ matrices, U and V are $m \times 1$ vector-columns. This realization is obtained by the reduction (via the change of variables) of the usual quadratic form in $\mathbb{C}^{2m+1}$ to the form defined by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}$, where I_m is the identity $m \times m$ matrix. In the new coordinates a function f is harmonic if and only if $(\partial/\partial x_1^2 + 2\sum_{i=1}^m \partial/\partial x_{i+1}\partial x_{m+i+1})f = 0$.

There is an embedding of g_2 to $\mathrm{so}(7, \mathbb{C})$ as the set of matrices

$$q = \begin{pmatrix} 0 & -w_1\sqrt{2} & -w_3\sqrt{2} & -z_4\sqrt{2} & -z_1\sqrt{2} & -z_3\sqrt{2} & -w_4\sqrt{2} \\ z_1\sqrt{2} & h_1 & w_2 & z_5 & 0 & z_4 & -w_3 \\ z_3\sqrt{2} & z_2 & h_2 & z_6 & -z_4 & 0 & w_1 \\ w_4\sqrt{2} & w_5 & w_6 & -h_1-h_2 & w_3 & -w_1 & 0 \\ w_1\sqrt{2} & 0 & w_4 & -z_3 & -h_1 & -z_2 & -w_5 \\ w_3\sqrt{2} & -w_4 & 0 & z_1 & -w_2 & -h_2 & -w_6 \\ z_4\sqrt{2} & z_3 & -z_1 & 0 & -z_5 & -z_6 & h_1+h_2 \end{pmatrix}$$

(see [19]), h_i corresponds to t, z_i corresponds to n_+ and w_j corresponds to n_-. Note that h_1 and h_2 are short roots of the Lie algebra g_2.

Proposition 2. *All* G_2*-invariant spaces on* S^6 *are* $\mathrm{SO}(7)$*-invariant.*

Proof. The harmonic polynomial x_7^k (x_7 is a coordinate function) is annihilated by $\rho(n_+)$. It is the highest vectors of the irreducible representation with the highest weight $k(h_1+h_2)$. By the Weyl formula (1) the dimension of the invariant space, generated by x_7^k, is equal to $(k+4)(2k+5)/5!k!$ and is equal to the dimension of H_k. It means that G_2-invariant spaces are $\mathrm{SO}(7)$-invariant.

It is convenient to realize $\mathrm{so}(2m, \mathbb{C})$ as the set of matrices $\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix}$, where X is arbitrary, Y and Z are skew-symmetric $m \times m$ matrices. This realization is obtained by the reduction of the usual quadratic form in $\mathbb{C}^{2m}$ to the form $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ by a linear change of variables. In the new coordinates a function f is harmonic if $(\sum_{i=1}^{m} \partial/\partial x_i \partial x_{m+i})f = 0$, where $\overline{x_i} = x_{m+i}$. To describe invariant algebras in the cases (8) and (9) we consider an embedding of so$(2l+1,\mathbb{C})$ to so$(2^l,\mathbb{C})$ corresponding to the spinor representation, details see in [8].

Let C^+ be a subalgebra of even elements of the Clifford algebra. Choose generating elements $v_1,\dots,v_l$, $w_1\dots,w_l$ of C^+ such that $v_iv_j + v_jv_i = 0$, $w_iw_j + w_jw_i = 0$, $v_iw_j + w_jv_i = -2\delta_{ij}$. A basis of C^+ is the set of elements $v_{i_1}\dots v_{i_r}w_{j_1}\dots w_{j_s}$ where $i_1 < \dots < i_r$, $j_1 < \dots < j_s$. The space generated by $v_1\dots v_l w_{j_1}\dots w_{j_s}$, $j_1 < \dots < j_s$, is 2^l-dimensional right ideal in C^+.

Let (h_i, e_i, f_i) be sl$(2,\mathbb{C})$-triple corresponding to the simple root α_i of the Lie algebra so$(2l+1,\mathbb{C})$. The elements h_i generate t, e_i generate n_+. The spinor representation τ of so$(2l+1,\mathbb{C})$ is defined by formulas

$\tau(e_i)z = \frac{1}{2}zv_iw_{i+1}$, $i = 1,\dots,l-1$; $\quad \tau(e_l)z = \frac{1}{2}zv_l$;

$\tau(h_i)z = \frac{1}{2}z(v_iw_i - v_{i+1}w_{i+1})$, $i = 1,\dots,l-1$; $\quad \tau(h_l)z = 1 + v_lw_l$.

Let $x_1 = v_1\dots v_l$ be a coordinate function. The harmonic polynomial x_1^k is the highest vector of the irreducible representation with the highest weight kh_l. The harmonic polynomial $x_{2^{l-1}+1}^k$ is the lowest vector of this representation, $x_{2^{l-1}+1} = v_1\dots v_lw_1\dots w_l$.

Proposition 3. *All* Spin$_7$*-invariant spaces on* S^7 *are* SO(8)*-invariant.*

Proof. The dimension of the irreducible representation of so$(7,\mathbb{C})$ with the highest weight kh_3 is equal to $(k+5)!(2k+6)/6!k!$ by the Weyl formula (1) and is equal to the dimension of H_k. It means that H_k is irrducible.

Proposition 4. *There are only five* Spin$_9$*-invariant algebras and they are self-adjoint: the three* SO(16)*-invariant algebras, the algebra of functions which are constant on the fibres of the Hopf fibration* $S^{15} \to S^8$ *and its subalgebra of functions which are even on the base of the fibration.*

Proof. Set

$$y = (v)(vw_2w_3w_4) - (vw_2)(vw_3w_4) + (vw_3)(vw_2w_4) - (vw_4)(vw_2w_3),$$

$$y_1 = x_9(vw_1) - (vw_2w_2)(vw_1w_3w_4) + (vw_1w_3)(vw_1w_2w_4) - (vw_1w_4)(vw_1w_2w_3),$$

where the elements of the Clifford algebra are contained in the brackets and $v = v_1v_2v_3v_4$, we multiply the brackets as coordinate functions. The harmonic polynomial $s_1 = x_1^ky^l$ is annihilated by $\rho(n_+)$, it is the highest vector of the irreducible representation with the highest weight $kh_4 + lh_1$. The lowest vector of this representation is $s_2 = x_9^ky_1^k = \overline{x_1^ky^l}$. Denote the corresponding invariant subspace of H_{k+2l} by $V_{k,l}$. It could be shown that $H_k \subset C(S^{15})$ is a direct sum of $V_{k-2i,i}$, $i = 0,\dots,[k/2]$ (see [19, pp. 304-305] with another notation).

Proposition 1 implies that the algebra B of functions which are constant on the fibres of the Hopf fibration Spin$_9$/Spin$_7 = S^{15} \to S^8 =$ Spin$_9$/Spin$_8$ and its subalgebra of functions which are even on the base of the fibration are invariant algebras, namely the closures of $\sum V_{0,l}$ and $\sum V_{0,2l}$. There are no other nontrivial SO(9)-invariant subalgebras of $B \simeq C(S^8)$.

Suppose that an invariant algebra A contains the space $V_{k,l}$, $k \geq 1$. Since $s_3 = \tau(e_4)s_2 = (vw_1w_2w_3)x_9^{k-1}y_1^k$ we have $\langle (vw_1w_2w_3)x_1, s_1s_3\rangle \neq 0$, i. e. $s_4 = (vw_1w_2w_3)x_1 \in A$. Projections of s_4 on the spaces $V_{2,0}$ and $V_{0,1}$ are non-zero, so all even functions are contained in A. If k is odd then A coincides with $C(S^{15})$.

2 Complex spheres

Let $O(p,q)$ be the space of homogeneous polynomials of degree p on z and q on $\overline{z}$, $H(p,q)$ be its subspace of harmonic polynomials. Let π be the representation of U(n) in O_k, $\pi(g)s(z) = s(g^{-1}z)$. The tangent representation of u(n) is defined by

$$\pi(X)s(z) = \frac{d}{dt}[s(\exp(-tX)z)]|_{t=0} = \{\nabla_z s(z), -Xz\} + \{\nabla_{\overline{z}} s(z), -\overline{Xz}\},$$

where $\nabla_y s = (\partial/\partial y_1,\dots\partial/\partial y_n)$, $\{a,b\} = \sum a_ib_i$.

Choose a basis of u(n) in the form $u_{jk} = e_{jk} - e_{kj}, j < k$; $\quad v_{jk} = i(e_{jk} + e_{kj})$; $\quad t_j = ie_{jj}$. Then

$$\pi(u_{jk}) = -(z_k\frac{\partial}{\partial z_j} - z_j\frac{\partial}{\partial z_k} + \overline{z_k}\frac{\partial}{\partial \overline{z_j}} - \overline{z_j}\frac{\partial}{\partial \overline{z_k}})s,$$

$$\pi(v_{jk}) = -i(z_k\frac{\partial}{\partial z_j} + z_j\frac{\partial}{\partial z_k} - \overline{z_k}\frac{\partial}{\partial \overline{z_j}} - \overline{z_j}\frac{\partial}{\partial \overline{z_k}})s,$$

$$\pi(t_j) = -i(z_j\frac{\partial}{\partial z_j} - \overline{z_j}\frac{\partial}{\partial \overline{z_j}})s.$$

Choose a basis of $\mathrm{u}(n)^{\mathbb{C}} = \mathrm{gl}(n,\mathbb{C})$ as $l_{jj} = -it_j = e_{jj}$, $l_{jk} = (u_{jk} - iv_{jk})/2 = e_{jk}$. We have $\pi(e_{jk}) = -z_k\partial/\partial z_j + \overline{z_j}\partial/\partial\overline{z_k}$.

It was shown in [17] that the spaces $H(p,q)$ are irreducible components of the representation of $\mathrm{U}(n)$. The Lie algebra $\mathrm{su}(n)^{\mathbb{C}} = \mathrm{sl}(n,\mathbb{C})$ consists of all matrices with zero trace, its subalgebra n_+ is generated by e_{ij}, $i < j$. The polynomial $z_n^p\overline{z_1}^q \in H(p,q)$ is annihilated by $\pi(n_+)$, so it is the highest vector of the irreducible representation. The operator $\sum_{k=1}^n e_{kk}$ acts on $H(p,q)$ as the multiplication on $(q-p)$ hence the spaces $H(p,q)$ are irreducible components of representation of $\mathrm{sl}(n,\mathbb{C})$. Moreover, if $n > 2$ then the highest vectors of different $H(p,q)$ have different eigenvalues under the action of $\pi(e_{11} - e_{22})$, i. e. $\mathrm{SU}(n)$-invariant spaces are $\mathrm{U}(n)$-invariant and we have

Proposition 5. *If $n > 2$ then $\mathrm{SU}(n)$-invariant algebras on S^{2n-1} are $\mathrm{U}(n)$-invariant.*

Since $\sum H(p,p)$ coincides with the set of all polynomials which are constant on all complex lines we have

Corollary. *All invariant algebras on $P^n\mathbb{C} = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$ are contained in the following list: $C(P^n\mathbb{C})$, $\mathbb{C}$ and (in the case $n = 1$) the algebra of functions which are constant on pairs of orthogonal complex lines.*

3 Quaternion spheres, the special cases

Let us consider the action of the group $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ on S^{4n-1}, $\mathrm{Sp}(n)$ acts by the multiplication from the left and $\mathrm{Sp}(1)$ acts by the multiplication from the right. We realize $\mathrm{Sp}(n)$ as the set of unitary $2n\times 2n$ matrices such that $S^tJS = J$, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Elements of $\mathrm{sp}(n,\mathbb{C})$ are matrices $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $B = B^t$, $C = C^t$, $D = -A^t$. Choose a basis of $\mathrm{sp}(n,\mathbb{C})$ in the form

$$a_{ij} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, b_{ij} = \begin{pmatrix} 0 & e_{ij}+e_{ji} \\ 0 & 0 \end{pmatrix}, a_{ij} = \begin{pmatrix} 0 & 0 \\ e_{ij}+e_{ji} & 0 \end{pmatrix}.$$

The Cartan subalgebra t of $\mathrm{sp}(n,\mathbb{C})$ is generated by a_{ii}, n_+ is generated by b_{ij} and a_{kl}, $k < l$.

Consider the restriction of the representation π to $\mathrm{sp}(n,\mathbb{C})$, $w_i = z_{n+i}$:

$$\pi(a_{ij}) = \pi(e_{ij}) - \pi(e_{n+j,n+i}) = -z_j\frac{\partial}{\partial z_i} + \overline{z_i}\frac{\partial}{\partial\overline{z_j}} + w_i\frac{\partial}{\partial w_j} - \overline{w_j}\frac{\partial}{\partial\overline{w_i}},$$

$$\pi(b_{ij}) = \pi(e_{i,n+j}) + \pi(e_{j,n+i}) = -w_j\frac{\partial}{\partial z_i} + \overline{z_i}\frac{\partial}{\partial\overline{w_j}} - w_i\frac{\partial}{\partial z_j} + \overline{z_j}\frac{\partial}{\partial\overline{w_i}},$$

$$\pi(_{ij}) = \pi(e_{n+i,j}) + \pi(e_{n+j,i}) = -z_j\frac{\partial}{\partial w_i} + \overline{w_i}\frac{\partial}{\partial\overline{z_j}} - z_i\frac{\partial}{\partial w_j} + \overline{w_j}\frac{\partial}{\partial\overline{z_i}}.$$

Since $\mathrm{sp}(1,\mathbb{C})\oplus\mathrm{sp}(1,\mathbb{C}) = \mathrm{so}(4,\mathbb{C})$ and $\mathrm{sp}(1)\oplus\mathrm{u}(1) = \mathrm{u}(2)$ we assume that $n \geq 2$. The polynomial

$$s_1 = w_1^p\overline{z}_1^q(w_1\overline{z}_2 - w_2\overline{z}_1)^r$$

is annihilated by $\pi(n_+)$; denote by $P(p,q,r)$ the corresponding invariant space. $P(p,q,r)$ is the space of the irreducible representation of $\mathrm{sp}(n,\mathbb{C})$ with the highest weight $(p+q+r, r, 0, \ldots, 0)$, the lowest vector of this representation is

$$s_2 = z_1^p\overline{w}_1^q(z_2\overline{w}_1 - z_1\overline{w}_2)^r.$$

It was shown in [10] that $H(p,q) = \oplus_{i=0}^{min(p,q)}P(p-i,q-i,i)$. Operators $L_1 = \{z,\nabla_{\overline{w}}\} - \{w,\nabla_{\overline{z}}\}$ and $L_2 = \{\overline{w},\nabla_z\} - \{\overline{z},\nabla_w\}$ commute with π (for every n). They map $P(p,q,r)$ on $P(p+1,q-1,r)$ and $P(p-1,q+1,r)$ respectively. Operators L_1, L_2 and $L_0 = [L_1,L_2] = \{w,\nabla_w\}+\{z,\nabla_z\}-\{\overline{w},\nabla_w\}-\{\overline{z},\nabla_z\}$ define the action of $\mathrm{Sp}(1)$. The spaces $\sum_{i=0}^{k-2l}P(i,k-i-2l,l)$, $0 \geq l \geq [k/2]$ are $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$-invariant irreducible subspaces of H_k.

Proposition 6. *All* $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$*-invariant algebras on* S^{4n-1} *are contained in the following list: the* $\mathrm{SO}(4n)$*-invariant algebras; the algebra of functions satisfying a condition* $f(w) = f(wq)$ *where* w *is a quaternion vector and* q *is an arbitrary quaternion from* $\mathrm{Sp}(1)$*; in the case* $n = 2$ *the algebra of functions satisfying the condition* $f(w) = f(z)$ *if* $\langle z, w\rangle = 0$*. All these algebras are self-adjoint.*

Proof. Closures of the spaces $\sum_{i=0}^{\infty} P(0,0,i)$ and $\sum_{i=0}^{\infty} P(0,0,2i)$ in the case $n = 2$ are the mentioned algebras. It is sufficient to show that there are no other ones.

Suppose that $P(p,q,r)$ and $P(k,l,m)$ lie in A, $p > q$ and $k < l$. Let s_1 and s_2 be the highest and the lowest vectors of representation in $P(p,q,r)$. Since $\langle (z_1w_1)^{p-q}, s_1s_2\rangle \neq 0$, $H(2p-2q,0) \subset A$. Similarly we have $H(0,2l-2k) \subset A$. It means that A contains $H(1,1)$ [17], in particular A contains $P(0,0,1)$. Therefore A contains $H(p+r,q+r)$ and $H(k+m,l+m)$, so A is $\mathrm{U}(2n)$-invariant and self-adjoint.. Moreover, if A is $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$-invariant then $P(2,0,0)$, $P(0,2,0)$, $P(1,1,0)$ and $P(0,0,1)$ are contained in A, the space of all even functions lies in A.

Suppose that an algebra A contains $P(0,0,l)$, $l > 0$, s_1 is the highest vector and s_2 is the the lowest vector of the representation in $P(0,0,l)$. $P(0,0,2)$ is contained in A because $\langle s_3, (\pi(c_{11})s_1)s_2\rangle \neq 0$, where s_3 is the lowest vector of representation in $P(0,0,2)$. If $n \geq 3$ then $P(0,0,1)$ is contained in the space generated by products of polynomials from $P(0,0,2)$ and the statement is proved. The exceptional algebra in the case $n = 2$ also could be described as the algebra of functions which are constant on the fibres of the Hopf fibration $S^7 \to S^4$ and even on the base of the fibration ($\mathrm{sp}(2,\mathbb{C}) = \mathrm{so}(5,\mathbb{C})$).

Corollary. *All* $\mathrm{Sp}(n+1)$*-invariant algebras on the quaternion projective space* $P^n\mathbb{H} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ *are contained in the following list:* $S(P^n\mathbb{H})$, $\mathbb{C}$ *and (in the case* $n = 1$*) the algebra of functions which are constant on pairs of orthogonal quaternion lines.*

Let us consider the action of the group $\mathrm{Sp}(n)\times\mathrm{U}(1)$ on S^{4n-1}. The generating element of $\mathrm{u}(1)$ acts on $P(p,q,r)$ by the multiplication on $(p-q)$. Therefore the spaces $P(p,q,r)$ are separated. It means that $P(p,q,r)$ is $\mathrm{Sp}(n)\times\mathrm{U}(1)$-invariant irreducible subspace of H_{p+q+2r}, $P(p,q,r)$ is a subspace of eigenfunctions of operators L_1L_2 and L_2L_1 with eigenvalues $(p+1)q$ and $(q+1)p$.

Proposition 7. *Every antisymmetric* $\mathrm{Sp}(n)\times\mathrm{U}(1)$*-invariant algebra on* S^{4n-1} *is a subalgebra of some antisymmetric* $\mathrm{U}(2n)$*-invariant algebra. The only* $\mathrm{Sp}(n)\times\mathrm{U}(1)$*-invariant self-adjoint algebra on* S^{4n-1} *which is not* $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ *and* $\mathrm{U}(2n)$*-invariant is the algebra of functions satisfying conditions* $f(w) = f(\alpha w)$, $|\alpha| = 1$ *and* $f(w) = f(wq)$*, where* $q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$*. Every* $\mathrm{Sp}(n)\times\mathrm{U}(1)$*-invariant algebra* A *could be represented as* $A = B \oplus S$ *where* B *is an antisymmetric invariant algebra and* S *is a self-adjoint invariant algebra.*

Proof. If $P(p,p,r)$ is contained in A then A contains the polynomial $s_1 = \overline{s_2}$, where s_1 is the highest vector and s_2 is the lowest vector, so A is not antisymmetric. If A is antisymmetric we may assume that A consists of $P(p,q,r)$, such that $p > q$ (the case of the alternative inequality corresponds to the conjugated algebra). Then A is contained in the antisymmetric $\mathrm{U}(2n)$-invariant algebra $\cup_{p>q} H(p,q) \cup H(0,0)$ and the first part of the proposition is proved.

Let A be a self-adjoint invariant algebra which is not $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$- and $\mathrm{U}(2n)$-invariant, as stated in the Proposition 6, $A \subseteq \sum_{k,l=0}^{\infty} P(k,k,l)$. Suppose that $P(k,k,l) \subset A$ and $k \neq 0$, s_1 and s_2 are the highest and the lowest vectors of the representation in $P(k,k,l)$. Since $\langle \pi(b_{22}s_3, (\pi(c_{22})s_1)s_2\rangle \neq 0$, where s_3 is the lowest vector of the representation in $P(0,0,2)$, $P(0,0,2)$ is contained in A. Since $\langle s_4, s_1^2 s_3^l\rangle \neq 0$, where s_4 is the highest vector of the representation in $P(2k,2k,0)$, $P(2k,2k,0)$ is contained in A too. Since $\langle \pi(b_{22}s_6, (\pi(c_{22})s_4)s_5\rangle \neq 0$, where s_5 is the lowest vector of the representation in $P(2k,2k,0)$ and s_6 is the lowest vector of the representation in $P(2,2,0)$, $P(2,2,0)$ is contained in A.

Union of all $P(k,k,l)$ with even k is the set of all polynomials satisfying conditions $f(w) = f(\alpha w)$, $|\alpha| = 1$ and $f(w) = f(wq)$ where $q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This invariant algebra could be described as the set of all polynomials which are constant on orthogonal complex lines lying on the same quaternion line.

For an invariant algebra A set $S = A \cap \bar{A}$. Then S is a self-adjoint invariant algebra. Let B be the orthogonal complement to S in A (in L^2), B consists of $P(p,q,r)$ such that either $p > q$ or $p < q$. If $f \in B$ and $h \in B$ then $fh \in B$. B is an antisymmetric invariant algebra and the proposition (and the theorem) is proved.

Corollary. *All invariant algebras on* $P^{2n+1}\mathbb{C} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times\mathrm{U}(1)$ *are contained in the following list:* $C(P^{2n+1}\mathbb{C})$*;* $\mathbb{C}$*; the algebra of even functions; the algebra of functions which are constant on the* *fibres* $\mathrm{Sp}(1)/\mathrm{U}(1)$ *of the fibration* $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times \mathrm{U}(1) \to \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\times \mathrm{Sp}(1)$ *and (in the case n=1) its subalgebra of functions which are even on the base of the fibration.*

4 Sp(1): a family of invariant algebras

We can identify the group $\mathrm{SU}(2) = \mathrm{Sp}(1)$ with the set of matrices $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$, $|a|^2 + |b|^2 = 1$ or with the sphere $S^3 \subset \mathbb{C}^2$ with the multiplication $(a,b)*(c,d) = (ac - \overline{b}d, bc + \overline{a}d)$.

It was shown in [10] that the set of all highest vectors of irreducible representations of $\mathrm{sp}(1,\mathbb{C})$ in H_k coincides with the set of polynomials of the form $\sum_{i=0}^{k} \gamma_i w^i \overline{z}^{k-i}$, $\gamma \in \mathbb{C}$.

The vector space generated by polynomials

$$a_1 = \alpha w^2 - w\overline{z},$$

$$a_2 = \pi(c_{11})a_1 = 2\alpha z w + w\overline{w} - z\overline{z},$$

$$a_3 = \frac{1}{2}\pi(c_{11})a_2 = \alpha z^2 + z\overline{w}$$

is invariant under the action of SU(2) from the left. Let A_α, $\alpha > 0$, be the invariant algebra with generating elements a_1, a_2, a_3 (the algebra $A_0 = \sum H(k,k)$ is U(2)-invariant and self-adjoint).

Proof of the Theorem 2. There are following relations between the generating elements:

$$a_2^2 - 4a_1a_3 = 1 \tag{2}$$

$$2|a_1|^2 + |a_2|^2 + 2|a_3|^2 = 1 + 2\alpha^2 \tag{3}$$

An image of the sphere under the mapping $T : \mathbb{C}^2 \to \mathbb{C}^3$ defined by the polynomials a_1, a_2 and a_3 is the set of points satisfying (2) and (3).

If p is a polynomial from A_α then p is a polynomial on a_1, a_2 and a_3 and satisfies $df \bigwedge da_1 \bigwedge da_2 \bigwedge da_3 = 0$. This equation is equivalent to

$$\left(z\frac{\partial}{\partial z} - (2\alpha z + \overline{w})\frac{\partial}{\partial \overline{w}} + w\frac{\partial}{\partial w} + (2\alpha w - \overline{z})\frac{\partial}{\partial \overline{z}}\right) f = 0 \tag{4}$$

Every operator commuting with π is some polynomial on L_0, L_1 and L_2. Vector fields iL_0, $L_1 - L_2$ and $i(L_1 + L_2)$ generate the space of all invariant real vector fields on the sphere. Vectors $dT(i(dz - d\overline{z}) + \alpha i(dw - d\overline{w}))$ and $dT(-\alpha(dw + d\overline{w}))$ generate a complex tangent line at the point $(0,-1,\alpha) = T(1,0)$. Since (4) is equivalent to

$$((i\alpha(L_1 + L_2) - iL_0) + i\alpha(L_1 - L_2))f = 0, \tag{5}$$

and at the point $(1,0)$ (5) gives

$$((i\alpha(\frac{\partial}{\partial \overline{w}} - \frac{\partial}{\partial w}) - i(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}})) + i\alpha(\frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}))f(1,0) = 0,$$

we obtain invariant CR-conditions on S^3.

Suppose that a homogeneous harmonic polynomial $p \neq$const is the highest vector of some irreducible component of the quasi-regular representation,

$$p = \sum_{i=0}^{k} \gamma_i w^i \overline{z}^{k-i}.$$

If p satisfies (4) we obtain the relations between γ_i:

$\gamma_0 = 0$, $(2i - k)\gamma_i + 2\alpha(k - 1 + i)\gamma_{i-1} = 0$.

They implies that $\gamma_i = 0$ for all $i < k/2$ and k is even. Moreover, p is uniquely determined by $\gamma_{k/2}$. It means that $p = \gamma_{k/2}a_1^{k/2}$, i. e. p lies in A_α. Since every polynomial on the sphere is the sum of homogeneous harmonic polynomials we prove the first part of the theorem.

For every operator $L = t_0 iL_0 + t_1L_1 + t_2L_2$ we can choose γ_i, $i = 1,2,3$, such that the polynomial $p = \gamma_0\overline{z}^2 + \gamma_1 w^1\overline{z}^1 + \gamma_2 w^2$ is annihilated by L. There is a right translation T such that $p*T = \gamma_0'\overline{z}^2 + \gamma_1' w^1\overline{z}^1$.

Then $L*T = t_0`iL_0 + t_1`L_1$. Since the usual CR-conditions are defined by the equation $L_1 f = 0$ suppose that $\gamma_1 = 1$ and $t_0 = 1$. The right translation by the corresponding diagonal matrix gives a_1 for some $\alpha \geq 0$. Note that the equation $L_0 f = 0$ doesn't define CR-structure, so the Theorem 2 is proved.

Proof of the Theorem 3. Suppose that there exists $f \in A_\alpha$ such that $\overline{f} = f$. Algebra A_α is contained in $\cup_{p\geq q\geq 0} H(p,q)$. Therefore f lies in the closure of $\cup_{p=0}^{\infty} H(p,p)$. If f is not constant function then $H(2,2)$ lies in A_α (see [17]). Since $z^2\overline{w}^2 \in H(2,2)$ does not satisfy (4) f is a constant function.

The maximal ideal space of the algebra A_α is the polynomially convex hull of the image of the sphere in $\mathbb{C}^3$ [1]. Let $z = \theta\zeta$, $w = \theta\eta$, $|\theta| = 1$, $|\zeta|^2 + |\eta|^2 = 1$. Then

$$a_1 = \alpha\theta^2\zeta^2 + \zeta\overline{\eta}, \quad a_2 = 2\alpha\theta^2\zeta\eta + |\eta|^2 - |\zeta|^2, \quad a_3 = \alpha\theta^2\eta^2 - \eta\overline{\zeta}.$$

This mapping extends holomorphically on θ in the unit disc by a natural way. A calculation shows that the family of mappings $f_{\zeta\eta} : \mathbb{D} \to \mathbb{C}^3$ covers a part of the hyperboloid (2) which is contained in the ellipsoid (3). Thus we have found M_α.

The algebra A_α consists of all analytic in the relative interior and continuous up to the boundary functions since it is generated by the analytic polynomials and every analytic fuction satisfies CR-conditions (4).

There is a transitive action ρ of $\mathrm{SU}(2)^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ on the hyperboloid (2), $\rho(T)M = TMT^t$, where $M = \begin{pmatrix} 2a_1 & a_2 \\ a_2 & 2a_3 \end{pmatrix}$, the emdedding of the sphere is equivariant. Choose three subgroups of SU(2):

$$G_1 = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix},\ G_2 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},\ G_3 = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}.$$

Points $\pm(0,1,0)$, $\pm 1/2(1,0,-1)$ and $\pm 1/2(i,0,i)$ are the only fixed points with respect to the action of G_1, G_2 and G_3 respectively. Hence SU(2) have no fixed points.

Let μ be a linear functional corresponding to the invariant normalized measure on S^3, precisely the Haar measure on the group SU(2). Then $\mu(a_1) = \mu(a_2) = \mu(a_3) = 0$ but $\mu(1) = 1$, so μ is not a multiplicative functional on A_α.

Suppose that $A_\alpha = \oplus\sum_{k\in Z_+} B_k$ is an invariant $\mathbb{Z}_+$-grading. It means that constant functions lie in B_0. The relation (2) implies that if B is the invariant space generated by a_1, a_2 and a_3 then $B^2 \subset B_0$. Hence A is contained in B_0, and the Theorem 3 is proved.

Remark. A minimal $\mathrm{Sp}(n)$-invariant algebras on the sphere in $\mathbb{C}^{2n}$, $n \geq 2$, containing the highest vector $\alpha w_1^2 - w_1\overline{z_1}$, is not antisymmetric. If $n = 2$ it contains the algebra of functions which are constant on all pairs of quaternion lines. If $n \geq 3$ it contains the algebra of functions which are constant on all quaternion lines.

References

[1] *Gamelin T.*. Uniform Algebras. Prentice-Hall, Englewood Cliffs, N. J., 1969.

[2] *Gangolli R.*. Invariant function algebras on compact semisimple Lie groups. Bull. Amer. Math. Soc., v.71, No 3, 634-637 (1965).

[3] *Gelfand I. M., Gindikin S. G.* . Complex manifolds whose skeletons are semi-simple real Lie groups, and analytic discrete series of representations. Funkt. Anal. i Pril., v. 11, 19-27 (1977) [Russian]; Funct. Anal. Appl., v. 11, 258-265 (1977).

[4] *Gichev V. M.*. Invariant function algebras on Lie groups. Sib. Mat. Zhurnal, v. 20, No 1, 23-36 (1979) [Russian].

[5] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres I. Deponir. in VINITI, 1615, (1980) [Russian].

[6] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres II. Deponir. in VINITI, 1616, (1980) [Russian].

[7] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres III. Deponir. in VINITI, 1617, (1980) [Russian].

[8] *Jacobson N.*. Lie Algebras. Wiley (Interscience), New York, 1962.

[9] *Kane J.* . Maximal ideal spaces of U-algebras. Illinois J. Math., v.27, No 1, 1-13 (1983).

[10] *Latypov I. A.*. Closed invariant spaces of functions on quaternion spheres. Deponir. in VINITI, 1461 (1996) [Russian].

[11] *Leeuw K. de, Mirkil H.*. Rotation-invariant algebras on the n-sphere. Duke Math. J., v. 30, No 4, 667-672 (1961).

[12] *Nagel A., Rudin W.*. Moebius-invariant function spaces on balls and spheres. Duke Math. J., v.43, No 4, 841-865 (1976).

[13] *Olshanskiĭ G. I.*. Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series. Funkt. Anal. i Pril., v. 15, 19-27 (1981) [Russian]; Funct. Anal. Appl., v. 15, 275-285 (1982).

[14] *Onishchik A. L.* . Transitive compact transformation groups. Mat. Sb., 60, 447-485 (1963) [Russian]; Am. Math. Soc. Transl., 55, 153-194 (1966).

[15] *Rider D.*. Translation-invariant Dirichlet algebras on compact groups. Proc. Amer. Math. Soc., v.17, No 5, 977-985 (1966).

[16] *Rosenberg A. L.*. Invariant algebras on compact groups. Mat. Sb., v. 81, No 2, 176-184 (1970) [Russian].

[17] *Rudin W.*. Function Theory in the Unit Ball of $\mathbb{C}^n$. Grundlehren Math. Wiss. 241, Springer, New York, 1982.

[18] *Wolf J.*. Translation-invariant function algebras on compact groups. Pacif. J. Math., v.15, No 3, 1093-1099 (1965).

[19] *Zhelobenko D. P., Stern A. I.*. Representations of Lie groups. Mir, Moscow, 1983 [Russian].