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Right multiplication operators in the clan structure of a Euclidean Jordan algebra

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§ 1. Preliminaries about Euclidean Jordan algebras

Vinberg's theory [3] tells us that associated to a homogeneous open convex cone containing no entire line, we have a clan structure in the ambient vector space. In this note we deal with the symmetric cone in a Euclidean Jordan algebra, and describe the associated clan structure.

Let V be a simple Euclidean Jordan algebra of rank r with unit element e. For $x \in V$, we denote by $M(x)$ the multiplication operator[1] by x, so that $M(x)y = xy$ for any $y \in V$. Let tr denote the trace function on the Jordan algebra V, and define an inner product in V by $\langle x|y\rangle := \operatorname{tr}(xy)$. Let us fix a Jordan frame $c_1, \dots, c_r$. We have $c_1 + \cdots + c_r = e$. The Jordan frame $c_1, \dots, c_r$ yields the Peirce decomposition $V = \bigoplus_{j \leqslant k} V_{jk}$, where $V_{jj} = \mathbb{R}c_j$ $(j = 1, \dots, r)$, and

$$V_{jk} := \left\{ x \in V \; ; \; M(c_i)\, x = \frac{1}{2}(\delta_{ij} + \delta_{ik})x \quad (i = 1, \dots, r) \right\} \quad (1 \leqslant j < k \leqslant r).$$

Let $\Omega := \operatorname{Int}\{x^2 \; ; \; x \in V\}$, the interior of squares in V, be the symmetric cone in V. The linear automorphism group of the cone Ω is denoted by $G(\Omega)$:

$$G(\Omega) := \{g \in GL(V) \; ; \; g(\Omega) = \Omega\}.$$

We know that $G(\Omega)$ is reductive. Let $\mathfrak{g}$ be the Lie algebra of $G(\Omega)$, and $\mathfrak{k}$ the derivation algebra $\operatorname{Der}(V)$ of the Jordan algebra V. Put $\mathfrak{p} := \{M(x) \; ; \; x \in V\}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with the corresponding Cartan involution $\theta X = -\,{}^tX$. Let

$$\mathfrak{a} := \mathbb{R}M(c_1) \oplus \ldots \oplus \mathbb{R}M(c_r).$$

[1] The notation in the book [1] is $L(x)$. Since we use left multiplication operators in clans, we have chosen a different symbol to avoid any confusion.

Then $\mathfrak{a}$ is an abelian subalgebra which is maximal in $\mathfrak{p}$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $M(c_1), \ldots, M(c_r)$. We know that the positive $\mathfrak{a}$-roots are $(\alpha_k - \alpha_j)/2$, $k > j$, and the corresponding root spaces $\mathfrak{g}_{(\alpha_k-\alpha_j)/2} =: \mathfrak{n}_{kj}$ are described as

$$\mathfrak{n}_{kj} := \{z \,\square\, c_j \; ; \; z \in V_{jk}\},$$

where $a \,\square\, b := M(ab) + [M(a), M(b)]$. Summing up all of the $\mathfrak{n}_{kj}$ as $\mathfrak{n} := \sum_{j<k} \mathfrak{n}_{kj}$, we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. Let $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$, the subgroups of $G(\Omega)$ corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively. The semidirect product group $H := N \rtimes A$ acts on Ω simply transitively, so that the orbit map $H \ni h \mapsto he \in \Omega$ is a diffeomorphism. Then its derivative at the unit element of H gives rise to a linear isomorphism $\mathfrak{h} := \mathrm{Lie}\,(H) \ni X \mapsto Xe \in V$. Its inverse map is denoted as $V \ni v \mapsto X_v \in \mathfrak{h}$. We have by definition $X_v e = v$.

§ 2. Clan structure of a Euclidean Jordan algebra

We keep to the notation in § 1. Let us introduce a bilinear product $\triangle$ in V by

$$v_1 \triangle v_2 := X_{v_1} v_2 \qquad (v_1, v_2 \in V).$$

By Vinberg [3], the product $\triangle$ defines a clan structure in V, that is, we have

(C1) $[X_{v_1}, X_{v_2}] = X_{v_1 \triangle v_2 - v_2 \triangle v_1}$ for all $v_1, v_2 \in V$;

(C2) there is $s \in V^*$ such that $\langle v_1 \triangle v_2, s\rangle$ defines an inner product in V;

(C3) the operators X_v $(v \in V)$ have only real eigenvalues.

We note that for (C2), it suffices to take $s = \mathrm{tr}(\cdot)$ in this case. For X_v we have the following lemma.

Lemma 2.1 (1) *If* $v = a_1 c_1 + \ldots + a_r c_r$ $(a_1 \in \mathbb{R}, \ldots, a_r \in \mathbb{R})$, *one has* $X_v = M(v)$.
(2) *If* $v \in V_{jk}$, *then* $X_v = 2(v \,\square\, c_j)$.

In what follows, we write R_v the right multiplication operator by $v \in V$:

$$R_v v' := v' \triangle v \qquad (v' \in V).$$

By noting that $c_1, \ldots, c_r$ are also primitive idempotents in the clan structure, the Peirce spaces V_{jk}, $j \leqslant k$, are the spaces for the normal decomposition relative to them:

$$V_{jk} = \{x \in V \; ; \; X_{c_i} x = \frac{1}{2}(\delta_{ij} + \delta_{ik})x, \, R_{c_i} x = \delta_{ij} x \quad (i = 1, \ldots, r)\}.$$

Thus the general clan multiplication rule is applied to the Peirce spaces, and we have

$$\begin{gathered} V_{kl} \triangle V_{jk} \subset V_{jl}, \\ \text{if } k \neq i, j, \text{ then } V_{kl} \triangle V_{ij} = 0, \\ V_{kl} \triangle V_{km} \subset V_{ml} \text{ or } V_{lm}, \text{ according to } m \geqslant l \text{ or } l \geqslant m. \end{gathered} \tag{2.1}$$

We put

$$\Xi := V_{1r} \oplus \ldots \oplus V_{r-1,r}, \qquad W := \Xi \oplus \mathbb{R}c_r.$$

Then (2.1) immediately implies

Proposition 2.2 *W is a two-sided ideal in the clan V. In other words, one has for any $v \in V$*

$$X_v(W) \subset W, \qquad R_v(W) \subset W.$$

In view of Proposition 2.2 we put for $v \in V$

$$R_v^W := R_v\big|_W.$$

Let us set

$$V' := \bigoplus_{1 \leqslant i \leqslant i \leqslant r-1} V_{ij}.$$

Then $V = V' \oplus W$, and V' is a Euclidean Jordan algebra of rank $r-1$, and thus has a clan structure. The right multiplication operator by $v' \in V'$ in the clan V' is denoted by $R'_{v'}$.

Corollary 2.3 *By writing $v \in V$ as $v = v' + w$ with $v' \in V'$ and $w \in W$, the operator R_v is of the form*

$$R_v = \begin{pmatrix} R'_{v'} & 0 \\ * & R_v^W \end{pmatrix}.$$

We next analyze the operator R_v^W. First, (2.1) implies that if $v' \in V'$, we have $R_{v'}(\Xi) \subset \Xi$. We put $R_{v'}^{\Xi} := R_{v'}\big|_{\Xi}$. To see what $R_{v'}^{\Xi}$ looks like, we define operators $\phi(v')$ $(v' \in V')$ on Ξ by

$$\phi(v')\xi := 2v'\xi \qquad (\xi \in \Xi).$$

Since V' (resp. Ξ) is the Peirce 0 (resp. the Peirce 1/2) space for the idempotent c_r, we know that the map $\phi : v' \mapsto \phi(v') \in \mathrm{End}(\Xi)$ is a unital Jordan algebra representation of V'. The following lemma is somewhat remarkable.

Proposition 2.4 *$R_{v'}^{\Xi} = \phi(v')$ for any $v' \in V'$.*

Proposition 2.5 *By writing $v \in V$ as $v = v' + \xi + v_r c_r$ with $v' \in V'$, $\xi \in \Xi$, $v_r \in \mathbb{R}$, the operator R_v^W is of the form*

$$R_v^W = \begin{pmatrix} \phi(v') & \dfrac{1}{2}\langle \cdot \,|\, c_r \rangle\, \xi \\ & \\ \langle \cdot \,|\, \xi \rangle\, c_r & v_r\, I_{V_{rr}} \end{pmatrix}.$$

We now renormalize the inner product $\langle \cdot | \cdot \rangle$ in $W = \Xi \oplus \mathbb{R}c_r$ by

$$\langle \eta + y_r c_r \,|\, \eta' + y'_r c_r \rangle_W := \langle \eta \,|\, \eta' \rangle + \frac{1}{2}\, y_r\, y'_r \qquad (\eta, \eta' \in \Xi \text{ and } y_r, y'_r \in \mathbb{R}).$$

Then the operator R_v^W expressed in Proposition 2.5 is written in a more symmetric way:

$$R_v^W = \begin{pmatrix} \phi(v') & \langle \cdot \,|\, c_r \rangle_W \, \xi \\ \langle \cdot \,|\, \xi \rangle_W \, c_r & v_r \, I_{V_{rr}} \end{pmatrix} \qquad (v = v' + \xi + v_r c_r). \tag{2.2}$$

In summary we obtain the following inductive structure for the right multiplication operators R_v:

$$R_v^W = \begin{pmatrix} \phi(v') & \frac{1}{2} \langle \cdot \,|\, c_r \rangle \, \xi \\ \langle \cdot \,|\, \xi \rangle \, c_r & v_r \, I_{V_{rr}} \end{pmatrix}.$$

Theorem 2.6 *Decomposing $v \in V$ as $v = v' + \xi + v_r c_r$ with $v' \in V'$, $\xi \in \Xi$ and $v_r \in \mathbb{R}$, one has*

$$R_v = \begin{pmatrix} R'_{v'} & 0 & 0 \\ * & \phi(v') & \langle \cdot \,|\, c_r \rangle_W \, \xi \\ * & \langle \cdot \,|\, \xi \rangle_W \, c_r & v_r \, I_{V_{rr}} \end{pmatrix}.$$

To get a "standard form" of the operator matrix R_v^W in (2.2), we first take $k \in \mathrm{Aut}(V')$ so that we have

$$v' = k(\lambda_1 c_1 + \ldots + \lambda_{r-1} c_{r-1}) \qquad (\lambda_1, \ldots, \lambda_{r-1} \in \mathbb{R}).$$

We have $\mathrm{Aut}(V') = \exp \mathrm{Der}(V')$, and we know that elements in $\mathrm{Der}(V')$ are all inner. Thus we write $k = \exp T'$, where T' is a sum of operators of the form $[M(a'), M(b')]$ with $a', b' \in V'$. In this way, we see that $\mathrm{Der}(V') \subset \mathrm{Der}(V)$, so that we have $\mathrm{Aut}(V') \subset \mathrm{Aut}(V)$. Hence we regard k as an element in $\mathrm{Aut}(V)$ such that $kc_r = c_r$ and $k\Xi \subset \Xi$. For $\eta \in \Xi$, we have

$$\begin{aligned} \phi(v')\eta &= 2v'\eta \\ &= 2k \left\{ (\lambda_1 c_1 + \ldots + \lambda_{r-1} c_{r-1})(k^{-1}\eta) \right\} \\ &= k \, (\lambda_1 P_1 + \ldots + \lambda_{r-1} P_{r-1}) \, k^{-1}\eta, \end{aligned}$$

where P_j denotes the orthogonal projection $\Xi \to V_{jr}$ $(j = 1, \ldots, r-1)$. Hence we obtain with $\xi' = k^{-1}\xi \in \Xi$,

$$R_v^W = k \begin{pmatrix} \lambda_1 P_1 + \ldots + \lambda_{r-1} P_{r-1} & \langle \cdot \,|\, c_r \rangle_W \, \xi' \\ \langle \cdot \,|\, \xi' \rangle_W \, c_r & v_r \, I_{V_{rr}} \end{pmatrix} k^{-1}. \tag{2.3}$$

Finally we compute $\det R_v$ $(v \in V)$ as an application of Theorem 2.6 and the expression (2.3). To do so, we recall the following obvious formula: if $\det A \neq 0$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

so that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det\left(D - CA^{-1}B\right).$$

We apply this to (2.3) under the condition that $\lambda_1 \cdots \lambda_{r-1} \neq 0$. Then, decomposing $\xi' \in \Xi$ as $\xi' = \xi'_1 + \ldots + \xi'_{r-1}$ with $\xi'_j \in V_{jr}$, we have by a simple computation

$$\begin{aligned} \det R_v^W &= (\lambda_1 \cdots \lambda_{r-1})^{d-1} \times \\ \times &\left\{\lambda_1 \cdots \lambda_{r-1} v_r - \frac{1}{2}\left(\lambda_2 \cdots \lambda_{r-1}\|\xi'_1\|^2 + \ldots + \lambda_1 \cdots \lambda_{r-2}\|\xi'_{r-1}\|^2\right)\right\}, \end{aligned} \quad (2.4)$$

where d is the common dimension of V_{jk}, $j < k$. Since both sides are polynomials in $\lambda_1, \ldots, \lambda_{r-1}$, the equality holds without the restriction $\lambda_1 \cdots \lambda_{r-1} \neq 0$. Now we have the following lemma gotten by applying [1, Proposition VI.3.2] to $c = c_1$.

Lemma 2.7 *One has*

$$\begin{aligned} &\Delta_r(\lambda_1 c_1 + \ldots + \lambda_{r-1} c_{r-1} + \xi' + v_r c_r) \\ = \ &\lambda_1 \cdots \lambda_{r-1} v_r - \frac{1}{2}\left(\lambda_2 \cdots \lambda_{r-1}\|\xi'_1\|^2 + \ldots + \lambda_1 \cdots \lambda_{r-2}\|\xi'_{r-1}\|^2\right). \end{aligned}$$

This lemma together with (2.4) shows that $\det R_v^W = \Delta_{r-1}(v)^{d-1}\Delta_r(v)$. Now summing up all the above discussions and using Theorem 2.6, we obtain by induction the following proposition:

Proposition 2.8 *For $v \in V$, one has* $\det R_v = \Delta_1(v)^d \cdots \Delta_{r-1}(v)^d \Delta_r(v)$.

Remark 2.9 We have $\det R_{hv} = \chi(h) \det R_v$, $h \in H$, $v \in V$, where $\chi(h) := (\det_V h)(\det \mathrm{Ad} h)^{-1}$. For this we refer the reader to the proof of [2, Lemma 2.7]. The one-dimensional representation χ of H comes from the linear form on $\mathfrak{a}$ given by $\sum_{j=1}^r [1 + d(r-j)]\alpha_j$. From this we also obtain Proposition 2.8.

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