Domains with homogeneous skeletons and invariant algebras

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Let N be a complex manifold, G be a real Lie group acting on N by holomorphic automorphisms, and $\mathcal{M}$ be a holomorphically convex domain in N whose skeleton is a single orbit $M = G/H$. In this article we consider the problem of a description of these domains in the case of compact G (the noncompact case is much more complicated). Because of the lack of a term, we shall call such domain *a semihomogeneous domain $\mathcal{M}$ over M*. Thus we fix the domain and the boundary — a domain may have several ideal boundaries.

There are many evident examples: the unit disc $\mathbb{D}$ and the upper half-plane $\mathbb{C}^+$ in $\mathbb{C}$, bounded symmetric domains and Siegel domains in $\mathbb{C}^n$. There are also "noncommutative" examples such as Olshanskiĭ semigroups (semigroups of the type $G\exp(iC)$ where G is a real form of a complex Lie group and C is an invariant cone in the Lie algebra $\mathcal{G}$ of G).

The set of all holomorphic and continuous up to the boundary functions on $\mathcal{M}$ is, for compact M, a closed subalgebra A of the Banach algebra $C(M)$, and the evaluating functional at any point of $\mathcal{M}$ defines a maximal ideal of A. Hence there is an embedding of $\mathcal{M}$ to the maximal ideal space $\mathcal{M}_A$ of A and the classification problem has a functional-analytic version: describe maximal ideal spaces of invariant algebras on homogeneous spaces of compact Lie groups. These problems are not quite equivalent; the second is, in certain sense, more natural because it often happens that all bounded holomorphic functions on $\mathcal{M}$ may be extended to some ideal analytic components at infinity. Furthermore, the functional-analytic approach makes it possible to apply the machinery of Harmonic Analysis and Banach algebras.

The problem has a solution for bi-invariant algebras (i.e., function algebras on groups invariant with respect to left and right shifts). For invariant algebras generated by finite dimensional invariant subspaces the problem is equivalent to the following one: describe polynomially convex hulls for orbits of compact linear groups. This subject is closely connected with actions of linear reductive groups or, in other words, with Invariant Theory.

Classical domains, especially the symmetric ones, are common fields of Complex Analysis, Harmonic Analysis, Representation Theory, Geometry, other mathematical disciplines. In this article only the geometrical part of the subject is considered.

Introduction

Siegel domains and their noncommutative analogues. *Siegel domains of the first kind* (an equivalent term — *tube domains*) are domains in $\mathbb{C}^n$ of the type $\mathbb{R}^n + i\,\mathrm{Int}(C)$ where C is a convex closed pointed generating cone in $\mathbb{R}^n$ and Int means "interior". To construct a *Siegel domain of the second type*, one needs spaces $\mathbb{C}^m$ and C^n, a cone $C \subset \mathbb{R}^n$ with the same properties, and a hermitian form h on $\mathbb{C}^m$ with values in $\mathbb{C}^n$ such that $h(z,z) \in C$ for all $z \in \mathbb{C}^m$ and $h(z,z) \neq 0$ for $z \neq 0$; the domain is defined as the interior of the set $\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \mathrm{Im}\, z - h(w,w) \in C\}$. In the both cases skeletons admit simply transitive groups of holomorphic automorphisms: translations in the first case (the skeleton is $\mathbb{R}^n$) and a two-step nilpotent group of affine transformations in the second one (the skeleton is the set $\mathrm{Im}\, z - h(w,w) = 0$). There is a generalization of Siegel domains of the second kind with a graded nilpotent Lie algebra instead of a two-step one ([21]).

Any bounded homogeneous domain in $\mathbb{C}^n$ is biholomorphically equivalent to a Siegel domain ([30]). Domains of the second kind are distinguished by the property to have a nontrivial CR-structure in skeletons, i.e. nontrivial complex linear subspaces in real tangent spaces (at the point $(0,0)$ this is the subspace $\{0\} \times \mathbb{C}^m$).

Olshanskiĭ semigroups (or *complex Lie semigroups*) are noncommutative analogues of Siegel domains of the first kind. Let G be a real form of a complex Lie group $G^{\mathbb{C}}$, $\mathcal{G}$ be the Lie algebra of G, and C be a convex closed pointed $\mathrm{Ad}(G)$-invariant cone in $\mathcal{G}$ (we shall say "invariant cone" for a cone with all these properties). Their structure is now well-understood. The set $G\exp(iC)$ is a subsemigroup of $G^{\mathbb{C}}$ and its interior is a semihomogeneous domain with the skeleton G. In a certain sense, Olshanskiĭ semigroups may exists even when the group G has no complexification; for example, this is true for the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. These semigroups have the property that representations of discrete holomorphic series extends to them and it was the reason for their consideration in [8] where a program for developing of Harmonic Analysis on them was introduced.

Noncommutative analogues of Siegel domains of the second type are less known. A simple example was recently pointed out by Latypov.

EXAMPLE 1. Let $\mathcal{M}$ be the intersection of a generic coadjoint orbit in $\mathrm{sl}(2,\mathbb{C})$ with the ball of radius r for r greater than the distance from the orbit to the origin (with respect to the $\mathrm{SU}(2)$-invariant hilbertian norm). Then $\mathcal{M}$ is a semihomogeneous domain whose skeleton is a single $\mathrm{SU}(2)$-orbit. The skeleton has a nontrivial CR-structure (this is easy to check by a comparison of dimensions); $\mathcal{M}$ is not holomorphically equivalent to a Siegel domain of the second kind because its automorphisms group coincides with $\mathrm{Ad}(\mathrm{SU}(2)) = \mathrm{SO}(3)$. The algebra A_r of all continuous in $\operatorname{clos}\mathcal{M}$ and analytic in $\mathcal{M}$ functions coincides with the uniform closure on $\mathcal{M}$ of the algebra of polynomials ([17]).

Invariant algebras. Invariant algebras were studied by many authors. In general, it was a part of attempts to extend the remarkable Function Theory in the Unit Disc $\mathbb{D}$ to several complex variables. It turns out that the multidimensional theory is quite different from one-dimensional; moreover, there is a significant difference between the Function Theory in the polydisc and in the ball. It was a reason for the consideration of various generalizations and the usage of additional tools such as Harmonic Analysis and Banach algebras. The setting of invariant algebras is a natural field for this machinery. We shall outline only one theme. The Wermer maximality theorem states that the disc-algebra (the algebra of analytic and continuous up to the boundary functions in $\mathbb{D}$) is a maximal subalgebra of $C(\mathbb{T})$ where $\mathbb{T}$ is the unit circle. A similar assertion for balls and polydiscs is not true; but it is true in the class of Möbius-invariant function algebras on the skeletons ([3],[14]; the Möbius group = group of holomorphic automorphisms). In fact, it is possible to give a complete list of these algebras for the ball ([20], see [28]). Here is an example of the usage of the results of this kind: suppose that a continuous in $\mathbb{D}$ function f has the property to have an analytic continuation from any circle of a fixed hyperbolic radius inside $\mathbb{D}$, then f is analytic in D. Proof: the set of such functions is a separating Möbius-invariant proper subalgebra of $C(\mathbb{D})$ closed in the topology of the uniform convergence on compact sets; by [27] (or [13]) it is either algebra of all analytic or algebra of all antianalytic functions, and the second case obviously cannot occur. The results of this type for balls are contained in ([28]). This book also contains a description of $\mathrm{U}(n)$-invariant algebras on spheres in $\mathbb{C}^n$. The maximal ideal spaces of these algebras described in [16] and [12] give examples of semihomogeneous domains of the first kind over spheres which are not equivalent to Siegel domains (see also the remark after the proof of Theorem 1).

Roughly speaking, if the situation is far from abelian then the family of invariant algebras is poor. For example, all bi-invariant algebras on semisimple compact groups are self-adjoint with respect to the complex conjugation; the same is true for $\mathrm{SO}(n)$-invariant algebras on spheres in $\mathbb{R}^n$ ([34], [7], [18]). If a bi-invariant algebra has no orthogonal real measures then the group is abelian ([25]). The abelian case, being almost trivial from the point of view of (noncommutative) Harmonic Analysis, is interesting for Function Theory. The set of characters of a compact abelian group G which are contained in an invariant algebra is a semigroup S and the maximal ideal space is isomorphic to the semigroup of homomorphisms of S to the multiplicative semigroup $\operatorname{clos}\mathbb{D}$ with pointwise multiplication and convergence. Here is an example of the algebras considered in [4], [19], [6], ch. 7, [15].

EXAMPLE 2. Let $S_\alpha = \{(n,m) \in \mathbb{Z}^2 : n + m\alpha \geq 0\}$ where $\alpha > 0$ is an irrational number. Then S_α is a semigroup, the maximal ideal space of the corresponding invariant algebra A_α on the dual group $\mathbb{T}^2$ is foliated by analytic half-planes whose boundary one-parametrical groups are dense windings of $\mathbb{T}^2$. This A_α is a maximal subalgebra of $C(\mathbb{T}^2)$ and may be realized as an algebra of analytic almost periodic functions on a half-plane (the uniform closure of linear combinations of exponents $e^{i(n+m\alpha)z}$, $n+m\alpha \geq 0$).

Conjectures, a problem, and two theorems

Bi-invariant algebras. The consideration of bi-invariant algebras in the paper [9] was based on the following observation: there is a natural structure of a compact topological semigroup in the maximal ideal space of an invariant algebra. The multiplication may be defined via the convolution of representing measures. The maximal ideal spaces of nontrivial bi-invariant algebras on a compact group G also admit a foliation by analytic leafs isomorphic to Olshanskiĭ semigroups. Only one such subsemigroup has the boundary containing the identity e of G.

The maximal ideal space $\mathcal{M}_A$ of a bi-invariant algebra A contains at most countable subsemigroup $\mathcal{I}$ consisting of pairwise commuting idempotents such that any idempotent in maximal ideal space is conjugated with some $\iota \in \mathcal{I}$. There is the zero $\iota_0 \in \mathcal{I}$. The algebra A is antisymmetric (a function algebra is called *antisymmetric* if it contains no real nonconstant functions) if and only if ι_0 is the zero of $\mathcal{M}_A$. Any $\iota \neq \iota_0$ corresponds some Olshanskiĭ subsemigroup. We shall call the semigroup corresponding to e *the main Olshanskiĭ semigroup.*

The author suppose to publish proofs of these results contained in hardly accessible articles [10],[11] in some of forthcoming papers.

A class of invariant algebras. Let H be a closed subgroup of a compact group G and $M = G/H$. The averaging by right over H (i.e. the operator $A_H f(g) = \int f(gh)\,dh$ where dh denotes the Haar measure on H) of any bi-invariant algebra on G gives an invariant algebra on M. Lets denote by $\mathfrak{A}$ the class of all invariant algebras which may be obtained by this way. Their maximal ideal spaces may be received by a kind of holomorphic projection from maximal ideal spaces of bi-invariant algebras. In other words, if $A \in \mathfrak{A}$ then there exists a semigroup (the maximal ideal space of a bi-invariant algebra on G) acting on $\mathcal{M}_A$ *transitively* (this will mean that the orbit of any point in M coincides with $\mathcal{M}_A$). Any $U(n)$-invariant algebra on the sphere in $\mathbb{C}^n$ belongs to $\mathfrak{A}$ ([12]) but algebras of Example 1 are not in $\mathfrak{A}$. Antisymmetric algebras of class $\mathfrak{A}$ has the property that G has a fixed point in the maximal ideal space (it corresponds to the zero of the maximal ideal space of the bi-invariant algebra).

Let A and $B \supseteq A$ be commutative Banach algebras. The embedding $A \to B$ induces the dual mapping $\mathcal{M}_B \to \mathcal{M}_A$. We shall say that A and B have the same maximal ideal spaces if this mapping is a bijection.

CONJECTURE 1. *For any invariant algebra A on $M = G/H$ which has a G-fixed point in $\mathcal{M}_A$ there exists an invariant algebra $B \in \mathfrak{A}$ on M which includes A and has the same maximal ideal space.*

If G is semisimple then any bi-invariant algebra on G is self-adjont. Thus the conjecture would imply that A has an additional group of symmetry.

Finitely generated invariant algebras and polynomially convex hulls. A Banach algebra A is generated by $a_1, \ldots, a_n \in A$ if the algebraically generated by $a_1, \ldots, a_n$ subalgebra is dense in A. If A is commutative then the mapping $\alpha : \varphi \to (\varphi(a_1), \ldots \varphi(a_n))$ is a homeomorphic embedding of $\mathcal{M}_A$ to $\mathbb{C}^n$. Further, if A is an uniform algebra on compact Q then $\alpha(\mathcal{M}_A) = \widehat{\alpha(Q)}$, where

$$\widehat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x \in X} |p(x| \quad \text{for all polynomials} \quad p\}$$

is the *polynomially convex hull of* $X \subseteq \mathbb{C}^n$.

We shall say that A is a *finitely generated invariant algebra* if it is generated as an uniform algebra by a finite dimensional invariant vector space $F \subseteq C(M)$. The evaluating functional at a base point $m \in M$ defines an equivariant embedding $\alpha : M \to F^*$ and $\mathcal{M}_A$ may be identified with $\widehat{\alpha(M)}$. This means that the problem of the description of maximal ideal spaces for finitely generated invariant algebras is equivalent to the following one.

PROBLEM. Describe polynomially convex hulls of orbits of compact linear groups.

The complexification $G^{\mathbb{C}}$ of G is a reductive algebraic linear group. There is a developed theory for actions of such groups ([31], [32]). Let $v \in V = F^*$, $\mathcal{O} = Gv$, and $\mathcal{O}^{\mathbb{C}} = G^{\mathbb{C}}v$. By the Hilbert–Mumford criterion, if $0 \in \operatorname{clos}(\mathcal{O}^{\mathbb{C}})$ then there exists an one-parametrical group γ in $G^{\mathbb{C}}$with the same property (i.e. $0 \in \operatorname{clos} \gamma v$). If $0 \notin \operatorname{clos}(\mathcal{O}^{\mathbb{C}})$ then there exists a $G^{\mathbb{C}}$-invariant polynomial which separates $\mathcal{O}^{\mathbb{C}}$ from zero. The set $\operatorname{clos}(\mathcal{O}^{\mathbb{C}})$ consists of a finite number of orbits and there is the unique closed one among them. The closure of an $G^{\mathbb{C}}$-orbit in the Zariski topology coincides with the closure in the real one. Hence

$$\widehat{\mathcal{O}} \subset \operatorname{clos}(\mathcal{O}^{\mathbb{C}}).$$

We may assume that there is no nontrivial G-fixed points in V. Then G has a fixed point in $\mathcal{M}_A$ if and only if $0 \in \operatorname{clos}(\mathcal{O}^{\mathbb{C}})$. Thus Conjecture 1 means that there exists a semigroup S of contractions of $\widehat{\mathcal{O}} = \mathcal{M}_A$ such that $Sv = \widehat{\mathcal{O}}$ (the action of G may be extended to the action of the maximal ideal space of a bi-invariant algebra; in general, the extension is not linear and the group is greater than G). This would be an analogue of the Hilbert-Mumford criterion for invariant algebras.

THEOREM 1. *Let ρ be a nontrivial irreducible representation of* $\mathrm{SL}(2,\mathbb{C})$ *in a finite dimensional complex linear space V, $G = \rho(\mathrm{SU}(2))$, $v \in V$, A be the closure of the algebra of polynomials in $C(\mathcal{O})$. If $0 \in \operatorname{clos}(\mathcal{O}^{\mathbb{C}})$ and the stabilizer of v in $G^{\mathbb{C}} = \rho(\mathrm{SL}(2,\mathbb{C}))$ is trivial then A admits an extension $B \in \mathfrak{A}$ with the same maximal ideal space; in other words, Conjecture* 1 *is true in this setting.*

Any invariant algebra may be approximated by finitely generated invariant algebras — it is the closure of an increasing sequence of them. However, their maximal ideal spaces may have new properties. Algebras A_α of Example 2 are not finitely generated ones and such effects as the irrational winding cannot occur for finitely generated invariant algebras. For a general antisymmetric bi-invariant algebra the main Olshanskiĭ semigroup need not be dense in $\mathcal{M}_A$ but for finitely generated invariant algebras it is always dense.

Invariant algebras without proper invariant ideals. Algebras of Example 1 has no proper invariant ideals. From the other hand, finitely generated invariant algebras of the class $\mathfrak{A}$ have many proper invariant ideals corresponding to the G-fixed point in $\mathcal{M}_A$(the maximal one and its powers).

CONJECTURE 2. *Let G, H, M be as above and A be a separating antisymmetric invariant algebra on M without proper invariant ideals. Then there exist a representation of G in a finite dimensional complex linear space V, a closed orbit $\mathcal{O}$ in V, and an equivariant embedding $M \to \mathcal{O}$ such that $\mathcal{M}_A = \widehat{\mathcal{O}}$. Furthermore, $\widehat{\mathcal{O}}$ contains the unique G-orbit M' such that the algebra $A|_{M'}$ is self-adjoint, there is an involutive antilinear automorphism of A commuting with G such that M' is exactly the set of all fixed points for the corresponding reflection in $\mathcal{M}_A$, and for any point $m' \in M'$ there exists a point $m \in M$ such that $m \in \widehat{H'm}$, where H' is the stabilizer of m'.*

In Latypov's example, M' is the nearest to zero $\mathrm{SU}(2)$-orbit in $\mathrm{SL}(2,\mathbb{C})$-orbit in $\mathrm{sl}(2,\mathbb{C})$. Let h, e, f be the sl_2-triple in the standard realization, $\mathcal{O}^{\mathbb{C}}$ and $\mathcal{O}$ be orbits of h for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SU}(2)$ respectively ($\mathcal{O}^{\mathbb{C}}$ is the set $\operatorname{Tr} Z^2 = 2$, the equation $\operatorname{Tr} Z^* Z = 2$ distinguishes $\mathcal{O}$ in $\mathcal{O}^{\mathbb{C}}$). The complex linear tangent space $\mathcal{T}$ at h to $\mathcal{O}^{\mathbb{C}}$ is $\mathbb{C}e + \mathbb{C}f$ and $(h + \mathcal{T}) \cap \mathcal{O}^{\mathbb{C}}$ is the union of two lines $h + \mathbb{C}e$ and $h + \mathbb{C}f$. The line $h + \mathbb{C}e$ intersects the ball $\operatorname{Tr} Z^* Z \leq 2 + r^2$ by the disc of radius r in $\mathbb{C}$ and the same is true for the second line. The homogeneous space M is $\mathrm{SU}(2)$-orbit of $h + re$; this is the intersection of $\mathcal{O}^{\mathbb{C}}$ and the sphere $\operatorname{Tr} Z^* Z = 2 + r^2$ (this set is connected). The reflection of the conjecture is $Z \to Z^*$. It transposes the two families of lines above. Further, $H' = \exp(i\mathbb{R}h)$, $H = \ker \operatorname{Ad} = \{1, -1\}$; for $m = h$ one may set $m' = h + re$ or $m' = h + rf$.

THEOREM 2. *Let G, $v \in V$, $\mathcal{O}$, and A be as in Theorem 1, and the stabilizer of v in $G^{\mathbb{C}}$ is connected. Then A has no proper invariant ideals if and only if $\mathcal{O}^{\mathbb{C}}$ is closed. In this case, either A is isomorphic to an invariant subalgebra of some algebra of Example* 1 *or $A = C(\mathcal{O})$.*

The following conjecture concerns the reconstruction problem for a general invariant algebra by the two opposite cases considered above.

CONJECTURE 3. *Any invariant algebra contains the unique maximal invariant ideal.*

This is an analogue of the following fact: the closure of an orbit of a reductive complex linear group contains the unique closed orbit.

Proof of Theorem 1. We use some facts of the theory of uniform algebras ([6], Ch. 3). There is the direct connection between the algebraic constructions and analytic ones; for example, the algebra of all bounded on $\mathcal{O}^{\mathbb{C}}$, holomorphic in the relative interior of $\widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}$ and continuous on $\widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}$ functions is the closure of the normalization of the algebra of polynomials restricted to $\mathcal{O}^{\mathbb{C}}$. The description of maximal ideal spaces if $\mathrm{U}(n)$invariant algebras on balls ([16]) has a great overlap with the classification of algebraic $\mathrm{SL}(2,\mathbb{C})$-embeddings (the description can be found in [32], Ch. 3, §4) — this fact seems to be not noted else.

If the closure of a three-dimensional $\mathrm{SL}(2,\mathbb{C})$-orbit $\mathcal{O}^{\mathbb{C}}$ contains 0 then it consists of three orbits: $\mathcal{O}^{\mathbb{C}}$, the two-dimensional orbit $\mathcal{O}^{\mathbb{C}}_h$ of the highest vector v_h, and $\{0\}$. By the Hilbert-Mumford criterion, there exists complex one-dimensional algebraic $\mathbb{C}$-torus ζ (i.e. homomorphism $\zeta : \mathbb{C}^* \to G^{\mathbb{C}}$ where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$)

such that $\lim_{z\to 0}\zeta(z)v = 0$. Let ζ_r be the corresponding one-parametrical multiplicative group of right shifts acting on $\mathcal{O}^{\mathbb{C}} = G^{\mathbb{C}}$. Since ζ_r commutes with G,

$$\lim_{z\to 0}\zeta_r(z)u = 0 \quad \text{for all} \quad u \in \mathcal{O}^{\mathbb{C}}. \tag{1}$$

In particular, this means that $0 \in \widehat{\mathcal{O}}$ because this mapping defines the embedding to $\widehat{\mathcal{O}}$ of the family $\zeta_r(\mathbb{D}\setminus\{0\})u \cup \{0\}$, $u \in \mathcal{O}$, of analytic discs with boundaries in $\mathcal{O}$. Moreover, (1) implies that $\zeta_r(z)(\widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}) \subseteq \widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}$ for all $z \in \mathbb{D}$, $z \neq 0$. Thus the action of the group $\tilde{G} = G \times \mathbb{T}$ (with $\mathbb{T} = \{\zeta_r(z) : |z| = 1\}$) on $\mathcal{O}$ is well-defined as well as the action of its complexification on $\mathcal{O}^{\mathbb{C}}$. This action extends continuously to $\operatorname{clos}\mathcal{O}^{\mathbb{C}}$ by the setting $\zeta_r|_{\mathcal{O}^{\mathbb{C}}_h} = \zeta|_{\mathcal{O}^{\mathbb{C}}_h}$

Let B be the closure in $C(\mathcal{O})$ of holomorphic in the relative interior of $\widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}$ and continuous on $\operatorname{clos}\mathcal{O}^{\mathbb{C}}$ functions on $\operatorname{clos}\mathcal{O}^{\mathbb{C}}$ which coincides with some function in A on $\mathcal{O}^{\mathbb{C}}_h$. We shall prove that Banach algebra B is the desired $\tilde{G}$-invariant extension of A and that $B \in \mathfrak{A}$. First off all, note that B is $\tilde{G}$-invariant. Indeed, the subalgebra B_0 of B consisting of functions vanishing on $\operatorname{clos}\mathcal{O}^{\mathbb{C}} \setminus \mathcal{O}^{\mathbb{C}}$ and holomorphic on $\mathcal{O}^{\mathbb{C}}$ is ζ_r-invariant because ζ_r continuously extends to $\operatorname{clos}\mathcal{O}^{\mathbb{C}}$; since ζ and ζ_r coincides on $\operatorname{clos}\mathcal{O}^{\mathbb{C}} \setminus \mathcal{O}^{\mathbb{C}}$, $f \circ \zeta - f \circ \zeta_r \in B_0$ for all $f \in B$, hence B is also ζ_r-invariant.

A function f on the maximal ideal space of an uniform algebra A is called *A-holomorphic* at the point $x \in \mathcal{M}_A$ if it can be uniformly approximated by functions in A on some neighborhood of x in $\mathcal{M}_A$; f is A-holomorphic on the set X if it is A-holomorphic at any point of this set. By [6], Ch. 3, §9, the following assertion holds: *if A is an uniform algebra and f is a continuous function on $\mathcal{M}_A$ which is A-holomorphic outside the set of its zeroes then the maximal ideal space and the Shilov boundary of the closed subalgebra of $C(\mathcal{M}_A)$ generated by A and f coincide with the maximal ideal space and the Shilov boundary of A respectively.* The hypothesis that B has an additional maximal ideal or that some function in B attains outside $\mathcal{O}$ a value which is greater than the uniform norm on $\mathcal{O}$ leads to a contradiction with this result. Indeed, suppose that there exist $f \in B$, $\varphi \in \mathcal{M}_B$, and $x \in \mathcal{M}_A = \widehat{\mathcal{O}}$ such that $\varphi(a) = a(x)$ for all $a \in A$ but $\varphi(f) \neq f(x)$. Since $B = \operatorname{clos}(A + B_0)$ we may assume that $f = a + b$ where $a \in A$ and $b \in B_0$. The uniform algebra B' on $\widehat{\mathcal{O}}$ generated by A and b has the same maximal ideal space because b is clearly A-holomorphic at any point x of $\mathcal{O}^{\mathbb{C}} \cap \widehat{\mathcal{O}}$ (it can be extended analytically to a neighborhood of x in V), and this is the contradiction. The assertion concerning the Shilov boundary is proved by the same arguments with the assumption $\sup\{|f(x)| : x \in \widehat{\mathcal{O}}\} > \sup\{|f(x)| : x \in \mathcal{O}\}$ instead of $\varphi(f) \neq f(x)$.

It remains to prove that $B \in \mathfrak{A}$. Lets consider the standard realization for representations of $\mathrm{SL}(2,\mathbb{C})$. The space V can be identified with the space of homogeneous polynomials of the degree $n = \dim V - 1$ of two complex variables z, w and $\mathrm{SL}(2,\mathbb{C})$ acts by $u(z,w) \to u(az + bw, cz + dw)$, $ad - bc = 1$. The assumption $\zeta(t)v \to 0$ as $t \to 0$ implies that, in suitable coordinates such that $\zeta(t)v(z,w) = v(tz, t^{-1}w)$,

$$v(z,w) = z^p w^q + c_1 z^{p+1} w^{q-1} \ldots + c_q z^n, \quad p + q = n, \quad p > q \tag{2}$$

Let k, l be positive integers, $k > l$. The mapping $\lambda_{k,l} : t \to v(t^l z, t^{-k} z + t^{-l} w)$ defines an analytic disc in $\mathcal{O}^{\mathbb{C}}$ for which the following conclusion holds:

$$lp = kq \iff \lim_{t\to 0}\lambda_{k,l}(t) = v_h \quad \text{and} \quad lp > kq \iff \lim_{t\to 0}\lambda_{k,l}(t) = 0 \tag{3}$$

where $v_h(z,w) = z^n$ is the highest vector. A direct calculation shows that the same is true for any g from the stabilizer of v_h and the disc $g\lambda_{k,l}(t)v$ (the stabilizer of z^n is the group $(z,w) \to (\varepsilon z, tz + \varepsilon^{-1}w)$ where $t \in \mathbb{C}$ and $\varepsilon^n = 1$). Let ρ_m be this representation in the space P_m of polynomials of the degree m and ι_r be the r-th power of the natural one-dimensional representation of $\mathbb{T}$. Any irreducible representation of the group $\tilde{G}$ has the form $\rho_{m,k} = \rho_m \otimes \iota_r$, with integer r and $m \in \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of nonnegative integers, and acts in the space P_m; if n even then m also must be even. The stationary subgroup $\sigma(t) = \zeta(t) \circ \zeta_r(-t)$, $|t| = 1$, of the point v acts on monomials by $\sigma(t) : z^k w^l \to t^{k-l-r} z^k w^l$. Hence it has a fixed point in P_m if and only if $|r| \leq m$ and $m - r$ is even and this is the monomial

$$v_{m,r}(z,w) = z^k w^l, \quad \text{where} \quad k = \frac{1}{2}(m + r), \; l = \frac{1}{2}(m - r). \tag{4}$$

The spaces $H(k,l)$ of matrix elements of the type $\langle \rho_{m,r}(g) v_{m,r}, \xi \rangle$, considered as functions on $\mathcal{O}$, defines the decomposition of the quasiregular representation of $\tilde{G}$ in $L^2(\mathcal{O})$ (the notation corresponds to the notation of [28], Ch.12).

By (2) and (3),

$$\lim_{t\to 0} f(\lambda_{p,q}(t)) = 0 \quad \text{for all} \quad f \in H(k,l) \iff \frac{l}{k} < \frac{q}{p}. \tag{5}$$

In other words, $H(k,l) \subset B_0$ if and only if $lp < kq$. Hence B_0 is the closed linear span of $H(k,l)$ where $k+l$ is even if n is even and $lp < kq$. Since for all $a = 1, \dots, q$ monomials $z^{p+a}w^{q-a}$ defines subspaces of B_0, it follows from (2) that B is generated by B_0 and $H(sp, sq)$, $s \in \mathbb{Z}^+$ (one can exclude these monomials consequently starting with z^n). Let $\mathcal{B}$ be the space on $\tilde{G}$ generated by all matrix element of representations $\rho_{m,r+a}$, where m, r are as in (4) with $lp < kq$, $a \in \mathbb{Z}^+$ (and is even if n is even), and $\rho_{s(p+q),s(p-q)}$, $s \in \mathbb{Z}^+$. Then the right averaging of $\mathcal{B}$ by σ gives B. This set of representations contains all irreducible components of their tensor products, hence the closure of $\mathcal{B}$ in $C(\tilde{G})$ is a bi-invariant algebra, and the proof is finished.

Remark. Since ζ_r is the group of right shifts, the action of $\tilde{G}$ on $\mathcal{O}$ may be identified with the action of U(2). It follows from [12] that $\widehat{\mathcal{O}} \cap \mathcal{O}^{\mathbb{C}}$ coincides with the orbit of the point v under the action of the semigroup

$$S_\alpha = \{M \in \mathrm{GL}(2) : ||M||||M^{-1}||^\alpha \le 1\}$$

where $\alpha = q/p$ and the norm is the operator norm with respect to the standard scalar product in $\mathbb{C}^2$. For any $\alpha \in [0,1)$ there exists an invariant algebra on the sphere $S^3 \subset \mathbb{C}^2$ whose maximal ideal space is the one-point compactification of an orbit of S_α in some linear space; these algebras are not finitely generated.

Proof of Theorem 2. If $\mathcal{O}^{\mathbb{C}}$ is not closed then it contains the unique closed orbit $\tilde{\mathcal{O}}$, and there exist polynomials vanishing on $\tilde{\mathcal{O}}$ but nontrivial on $\mathcal{O}$. Conversely, the set of common zeros of polynomials in the proper G-invariant ideal is $G^{\mathbb{C}}$-invariant and closed; since $\widehat{\mathcal{O}} \subset \operatorname{clos} \mathcal{O}^{\mathbb{C}}$, $\mathcal{O}^{\mathbb{C}}$ cannot be closed in this case.

If $\mathcal{O}^{\mathbb{C}}$ is closed and two-dimensional then either the real dimension of $\mathcal{O}$ is 2 or $\mathcal{O}$ has a nontrivial complex line in the tangent space to any point of $\mathcal{O}$. In the first case $\mathcal{O}$ is the two dimensional sphere or real projective plane and $A = C(\mathcal{O})$ by [18]. In the second one, functions in A satisfies some invariant CR-conditions. By [17], A is isomorphic to some subalgebra Example 1 or to a subalgebra of the algebra of analytic functions in the unit ball in $\mathbb{C}^2$. The last possibility cannot occur because G has a fixed point in $\mathcal{M}_A$ whence A contains a nontrivial invariant ideal.

Let $\mathcal{O}^{\mathbb{C}}$ be three-dimensional. Then the stabilizer of a point $v \in \mathcal{O}^{\mathbb{C}}$ is trivial by the assumption of the theorem, $\mathcal{O}^{\mathbb{C}}$ may be identified with $G^{\mathbb{C}}$ and $\mathcal{O}$ with its maximal compact subgroup G. Hence there exists an antiholomorphic involution in $\mathcal{O}^{\mathbb{C}}$ commuting with G with $\mathcal{O}$ as the set of all its fixed points. This means that the restriction to $\mathcal{O}$ of the algebra of all analytic in $\mathcal{O}^{\mathbb{C}}$ functions is self-adjoint with respect to the complex conjugation. Since $\mathcal{O}^{\mathbb{C}}$ is closed and smooth, any analytic in $\mathcal{O}^{\mathbb{C}}$ function may be extended analytically to a neighborhood of $\widehat{\mathcal{O}}$ in V. It remains to use the following well-known fact from the approximation theory: any analytic in a neighborhood of a polynomially convex compact set K function can be approximated by polynomials, and to finish the proof with the Stone-Weierstrass theorem.

References

[1] *Agranovsky M. L.*. Invariant algebras on boundaries of symmetric domains. Dokl. Akad. Nauk. v. 197, No 1, 9-11 (1971) [Russian].

[2] *Agranovsky M. L.*. Invariant algebras on Riemannian symmetric spaces. Dokl. Akad. Nauk, v. 207, No 3, 513-516 (1972) [Russian].

[3] *Agranovsky M. L., Valsky R. E.*. Maximality of invariant function algebras. Sib. Mat. Zhurnal, v. 12, No 1, 3-12 (1971) [Russian].

[4] *Arens R., Singer I. M.*. Generalized analytic functions. Transl. Amer. Mat. Soc., v. 81, No 2, 379-393 (1956).

[5] *Bjork J.-E.*. Compact group operating on Banach algebra. Math. Ann., v. 205, No. 4, 281-297 (1973).

[6] *Gamelin T.*. Uniform Algebras. Prentice-Hall, Englewood Cliffs, N. J., 1969.

[7] *Gangolli R.*. Invariant function algebras on compact semisimple Lie groups. Bull. Amer. Math. Soc., v.71, No 3, 634-637 (1965).

[8] *Gelfand I. M., Gindikin S. G.* . Complex manifolds whose skeletons are semi-simple real Lie groups, and analytic discrete series of representations. Funkt. Anal. i Pril., v. 11, 19-27 (1977) [Russian]; Funct. Anal. Appl., v. 11, 258-265 (1977).

[9] *Gichev V. M.*. Invariant function algebras on Lie groups. Sib. Mat. Zhurnal, v. 20, No 1, 23-36 (1979) [Russian].

[10] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres I. Deponir. in VINITI, 1615, (1980) [Russian].

[11] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres II. Deponir. in VINITI, 1616, (1980) [Russian].

[12] *Gichev V. M.*. Maximal ideal spaces of invariant algebras on compact Lie groups and spheres III. Deponir. in VINITI, 1617, (1980) [Russian].

[13] *Gichev V. M.*. On invariant algebras of continuous functions on balls and euclidean spaces. Sib. Mat. Zurn., v. 25 (1984), no. 4, p. 32–36.

[14] *Gorin E. A., Zolotarevsky V. M.*. Maximal invariant subalgebras of algebras with involution. Mat. Sb., v. 85, No 3, 373-387 (1971) [Russian].

[15] *Grigorian S. A.*. Generalized analytic functions. Uspehi Mat. Nauk, v. 49, No 2, 3-42 (1994) [Russian].

[16] *Kane J.* . Maximal ideal spaces of U-algebras. Illinois J. Math., v.27, No 1, 1-13 (1983).

[17] *Latypov I. A.*. Invariant algebras on spheres. This volume, p.??–??

[18] *Leeuw K. de, Mirkil H.*. Rotation-invariant algebras on the n-sphere. Duke Math. J., v. 30, No 4, 667-672 (1961).

[19] *Leeuw K. de, Mirkil H.*. Translation-invariant function algebras on abelian groups. Bull Soc. Math. France, v. 88, 345-370 (1960).

[20] *Nagel A., Rudin W.*. Moebius-invariant function spaces on balls and spheres. Duke Math. J., v.43, No 4, 841-865 (1976).

[21] *Naruki I.*. Holomorphic Extension Problem for Standard Real Submanifolds of Second Kind. Publ. RIMS Kyoto Univ, v. 6, 113-187 (1970).

[22] *Olshanskiĭ G. I.*. Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series. Funkt. Anal. i Pril., v. 15, 19-27 (1981) [Russian]; Funct. Anal. Appl., v. 15, 275-285 (1982).

[23] *Olshanskiĭ G. I.*. Complex Lie semigroups, Hardy spaces and the Gelfand-Gindikin program. In "Topics in group theory and homological algebra", Yaroslavl University Press, 85-98 (1982) [Russian]; Differential Geometry and its applications, v. 1, 2979-308 (1991) [English].

[24] *Pyatetskiĭ-Shapiro I.I.* . Geometry of classical domains and theory of automorphic functions. Gos. Izd. Fiz.-Mat. Lit., Moscow, 1961 [Russian].

[25] *Rider D.*. Translation-invariant Dirichlet algebras on compact groups. Proc. Amer. Math. Soc., v.17, No 5, 977-985 (1966).

[26] *Rosenberg A. L.*. Invariant algebras on compact groups. Mat. Sb., v. 81, No 2, 176-184 (1970) [Russian].

[27] *Rudin W.*. Möbius-invariant algebras in balls. Ann. Inst. Fourier, v. 33 (1983), no. 2, p. 19–41. Function Theory in the Unit Ball of $\mathbb{C}^n$. Grundlehren Math. Wiss. 241, Springer, New York, 1982.

[28] *Rudin W.*. Function Theory in the Unit Ball of $\mathbb{C}^n$. Grundlehren Math. Wiss. 241, Springer, New York, 1982.

[29] *Rudin W.*. Function Theory in polydiscs. Mathematics Lectrure Note Series, University of Wisconsin W. A. Benjamin, Inc. New York, 1969.

[30] *Vinberg E. B., Gindikin S. G., Pyatetskiĭ-Shapiro I.I.*. On classification and canonical realization for complex bounded homogeneous domains. Trudy Mosk. Mat. Ob., v. 12 (1963), p. 359–388 [Russian].

[31] *Vinberg E. B., Popov V. L.*. Invariant Theory. Encyclopedia Math., v. 55, 137-309, VINITI, 1990 [Russian]. [Russian].

[32] *Kraft H.*. Geometric methods in Invariant Theory. Mir, Moskow, 1987 (Translation to Russian of *Geometrishe Methoden in der Invariantentheorie*, Friedr Viewg&Sohnn, Braunschweig/Wiesbaden, 1985).

[33] *Shilov G. E.*. Homogeneous rings of functions. Uspehi Mat. Nauk, 6, No 1, 89-135 (1951) [Russian].

[34] *Wolf J.*. Translation-invariant function algebras on compact groups. Pacif. J. Math., v.15, No 3, 1093-1099 (1965).