MCS 22E47

Invariant finite-dimensional spaces of functions on two-dimensional algebras [1]

© **D. S. Tugaryov**
S.-Petersburg Acad. University, S.-Petersburg, Russia

A description of finite dimensional spaces of functions on algebras of generalized complex numbers invariant with respect to a motion group is presented

Keywords: two-dimensional algebras, groups of motions, Laplace operator

Let $\mathcal{A}$ be an algebra of generalized complex numbers $z = x + iy$, where $x, y \in \mathbb{R}$, with a relation $i^2 = \alpha + 2\beta i$. It is isomorphic to one of three algebras $\mathbb{C}$, $\mathbb{D}$, Λ: the algebra of complex ($i^2 = -1$), double ($i^2 = 1$) and dual ($i^2 = 0$) numbers respectively. For a number $z = x + iy$, the number $\overline{z} = x - iy$ is called conjugated to z. As coordinates on $\mathcal{A}$ one can also take z and $\overline{z}$.

The exponential function e^z is defined as the sum of a series:

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,.$$

Let G be the group of "motions" of the algebra $\mathcal{A}$: it is generated by parallel translations $z \mapsto z + a$, $a \in \mathcal{A}$, and "rotations" $z \mapsto \mathrm{e}^{it} z$, $t \in \mathbb{R}$.

The Laplace operator

$$\Delta = \frac{\partial^2}{\partial z \partial \overline{z}}$$

is invariant with respect to the group G. In coordinates x, y on algebras $\mathbb{C}$, $\mathbb{D}$, Λ the Laplace operator is relatively

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \quad \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\,, \quad \frac{\partial^2}{\partial x^2}\,.$$

For the algebra Λ, an invariant operator is also $\Delta_1 = \partial / \partial x$.

[1]Supported by the Russian Foundation for Basic Research (RFBR): grant 09-01-00325-a, Sci. Progr. "Development of Scientific Potential of Higher School": project 1.1.2/9191, Fed. Object Progr. 14.740.11.0349 and Templan 1.5.07

Let us describe *finite dimensional* spaces V of functions $f \in C^\infty(\mathcal{A})$, invariant with respect to G.

A space V, invariant with respect to G, is called *indecomposable*, if it cannot be represented as a direct sum of subspaces V_1 и V_2, invariant with respect to G.

Let us call an indecomposable space V, invariant with respect to G, *weakly indecomposable*, if there are no subspaces V_1 and V_2 of V, invariant with respect to G, such that $V/V_0 = V_1/V_0 + V_2/V_0$ where $V_0 = V_1 \cap V_2$.

Theorem 1.1 *Let $\mathcal{A}$ be $\mathbb{C}$ or $\mathbb{D}$. Any finite dimensional weakly indecomposable invariant with respect to G space V consists of polynomials $f(z,\overline{z})$ of degree $\leqslant k$ in z and of degree $\leqslant m$ in $\overline{z}$. Its dimension is equal to $(k+1)(m+1)$.*

Thus, the space V is the space of solutions of a system of equations:

$$\left(\frac{\partial}{\partial z}\right)^{k+1} f = 0, \ \left(\frac{\partial}{\partial \overline{z}}\right)^{m+1} f = 0.$$

Let us take a basis in V consisting of monomials $z^r\overline{z}^s$, $r \leqslant k$, $s \leqslant m$. In this basis, the Laplace operator Δ has a Jordan normal form, the number of Jordan boxes is equal to $k+m+1$. A basis for a Jordan box is formed by monomials $z^r\overline{z}^s$ with fixed difference $r-s$. For all boxes, their eigenvalues are equal to zero.

Theorem 1.2 *Let $\mathcal{A} = \Lambda$. Any finite dimensional weakly indecomposable invariant with respect to G space V consists of polynomials $f(x,y)$ of degree $\leqslant m$ in y and of degree $\leqslant k+m$ in x,y together. Its dimension is equal to $(k+1+m/2)(m+1)$.*

Thus, the space V is the space of solutions of a system of equations:

$$\left(\frac{\partial}{\partial y}\right)^{m+1} f = 0, \ \ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{k+m+1} f = 0.$$

Let us take a basis in V consisting of monomials x^ry^s, $r+s \leqslant k+m$, $s \leqslant m$. In this basis, the operator Δ_1 has a Jordan normal form. A basis for a Jordan box is formed by monomials x^ry^s with s fixed, the number of Jordan boxes is equal to $m+1$. For all boxes, their eigenvalues are equal to zero.