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Invariants for multiplicity-free actions

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Let a connected compact Lie group K act on a finite-dimensional vector space V over $\mathbb{C}$ linearly. Then K acts on the ring $\mathcal{P}(V)$ of holomorphic polynomials naturally. It is important to know when such an action is multiplicity-free. In particular, it is true for the action of K on $\mathfrak{p}_-$ for a non-compact Hermitian symmetric space G/K. We present four examples of multiplicity-free actions that are not weakly equivalent to any action of Hermitian type.

Keywords: compact Lie groups, rings of polynomials, multiplicity-free actions, invariants

Let V be a finite-dimensional vector space over $\mathbb{C}$, K a connected compact Lie group acting on V linearly. Then K acts on the ring of (holomorphic) polynomials $\mathcal{P}(V)$ naturally. The action (K, V) is *multiplicity-free* if each irreducible K-module appears in $\mathcal{P}(V)$ at most one. In this paper we describe K-invariant polynomials and K-invariant differential operators for certain multiplicity-free actions.

Take a K-invariant inner product $(\cdot,\cdot)_V$ on V. Then we can take the Fischer inner product $(\cdot,\cdot)_{\mathcal{F}}$ on $\mathcal{P}(V)$ given by

$$(f,g)_{\mathcal{F}} = \frac{1}{\pi^{\dim V}} \int_V f(z)\,\overline{g(z)}\,e^{-\|z\|_V^2}\,d\mu(z),$$

where μ is the Lebesgue measure on $V \simeq \mathbb{R}^{2\dim V}$.

Let $\overline{\mathcal{P}(V)}$ and $\mathcal{PD}(V)$ be the ring of anti-holomorphic polynomials and the ring of differential operators on V with polynomial coefficients, respectively. The group K acts also on $\overline{\mathcal{P}(V)}$ and $\mathcal{PD}(V)$. We denote by $\left(\mathcal{P}(V)\otimes\overline{\mathcal{P}(V)}\right)^K$ and by $\mathcal{PD}(V)^K$ the subrings of K-invariants, respectively.

Each irreducible K-module P_λ is determined by the highest weight λ. Denote by Λ the set of highest weights λ for $\mathcal{P}(V)$. The element $\lambda \in \Lambda$ is *fundamental* if the highest weight vector h_λ corresponding to λ is irreducible (as a polynomial). Let Λ_0 be the set of fundamental highest weights in Λ. Then Λ is a free Abelian semigroup with a basis Λ_0. We call the number of elements of Λ_0 the *rank* of (K, V).

For $\lambda \in \Lambda$ we take an orthonormal basis $\{f_1, \dots, f_{d_\lambda}\}$ for K-submodule P_λ, where $d_\lambda = \dim P_\lambda$. Then we have a K-invariant polynomial $p_\lambda(z,\overline{z})$ and K-invariant differential operator $p_\lambda(z,\partial)$ given by

$$p_\lambda(z,\overline{z}) = \sum_{j=1}^{d_\lambda} f_j(z)\overline{f_j(z)},$$

$$p_\lambda(z,\partial) = \sum_{j=1}^{d_\lambda} f_j(z)\overline{f}_j(\partial),$$

where $\overline{f}(\partial) = \sum_\alpha \overline{a}_\alpha \partial^\alpha$ for $f(z) = \sum_\alpha a_\alpha z^\alpha$.

The following theorem is given by Howe and Umeda.

Theorem 1 (Howe-Umeda) *Let $\lambda \in \Lambda_0$. Polynomials $p_\lambda(z,\overline{z})$ and differential operators $p_\lambda(z,\partial)$ are algebraically independent and generate $\left(\mathcal{P}(V)\otimes\overline{\mathcal{P}(V)}\right)^K$ and $\mathcal{PD}(V)^K$, respectively.*

For each irreducible K-submodule P_λ of $\mathcal{P}(V)$, the K-invariant differential operator $p_\lambda(z,\partial)$ preserves any irreducible K-submodule P_μ and acts on P_μ as a scalar multiple operator, that is, there exists $c_{\lambda,\mu} \in \mathbb{C}$ such that $p_\lambda(z,\partial)|_{P_\mu} = c_{\lambda,\mu}\mathrm{Id}_{P_\mu}$. We call $c_{\lambda,\mu}$ a *binomial coefficient*.

The typical multiplicity-free action is derived from a non-compact Hermitian symmetric pair (G,K). Let $\mathfrak{g}$, $\mathfrak{k}$ be Lie algebras of G, K, respectively. We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$. Then we have a decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_+ + \mathfrak{p}_-$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. The action $(K,\mathfrak{p}_-)$ of K on $\mathfrak{p}_-$ is multiplicity-free and the rank of $(K,\mathfrak{p}_-)$ is equal to the real rank of G. We call $(K,\mathfrak{p}_-)$ the action of *Hermitian type*. If (K,V) is of Hermitian type, the K-invariant polynomial $p_\lambda(z,\overline{z})$ and the binomial coefficient $c_{\lambda,\mu}$ are described with a Jack polynomial and a shifted Jack polynomial.

Let (K_1,V_1), (K_2,V_2) be multiplicity-free actions. We say that (K_1,V_1) is *weakly equivalent* to (K_2,V_2) if there exists a linear isomorphism $\phi : V_1 \longrightarrow V_2$ such that $\phi(K_1\cdot v) = K_2\cdot\phi(v)$ for any $v \in V_1$. If (K_1,V_1) is weakly equivalent to (K_2,V_2), then $\left(\mathcal{P}(V_1)\otimes\overline{\mathcal{P}(V_1)}\right)^{K_1}$ and $\mathcal{PD}(V_1)^{K_1}$ are isomorphic to $\left(\mathcal{P}(V_2)\otimes\overline{\mathcal{P}(V_2)}\right)^{K_2}$ and $\mathcal{PD}(V_2)^{K_2}$, respectively.

For two multiplicity-free actions (K_1,V_1), (K_2,V_2) we see that $K_1\times K_2$ acts on $V_1\oplus V_2$ naturally, and the action $(K_1\times K_2, V_1\oplus V_2)$ is multiplicity-free, too. The action $(K_1\times K_2, V_1\oplus V_2)$ is called the *product* of (K_1,V_1) and (K_2,V_2). In this case,

$$\left(\mathcal{P}(V_1\oplus V_2)\otimes\overline{\mathcal{P}(V_1\oplus V_2)}\right)^{K_1\times K_2} \simeq \left(\mathcal{P}(V_1)\otimes\overline{\mathcal{P}(V_1)}\right)^{K_1} \otimes \left(\mathcal{P}(V_2)\otimes\overline{\mathcal{P}(V_2)}\right)^{K_2},$$
$$\mathcal{PD}(V_1\oplus V_2)^{K_1\times K_2} \simeq \mathcal{PD}(V_1)^{K_1}\otimes\mathcal{PD}(V_2)^{K_2}.$$

The action (K,V) is *indecomposable* if there are no non-trivial actions (K_1,V_1), (K_2,V_2) such that (K,V) is the product of them.

Each indecomposable multiplicity-free action (K,V) of rank one or of rank two is weakly equivalent to some action $(K',\mathfrak{p}_-)$ of Hermitian type. In this paper, we treat the following actions.
(1) $(\mathbb{T}\times \mathrm{Sp}(n)\times \mathrm{SU}(2), \mathrm{Mat}(2n,2,\mathbb{C}))$, $n\geqslant 2$,
(2) $(\mathbb{T}^2\times \mathrm{SU}(n), \mathbb{C}^n\oplus\mathbb{C}^n)$, $n\geqslant 2$,
(3) $(\mathbb{T}\times \mathrm{SU}(2)\times \mathrm{SU}(n), \mathbb{C}^2\oplus \mathrm{Mat}(2,n,\mathbb{C}))$, $n\geqslant 2$,
(4) $(\mathbb{T}^2\times \mathrm{Sp}(n), \mathbb{C}^{2n}\oplus\mathbb{C}^{2n})$, $n\geqslant 2$.
These actions are not weakly equivalent to any action $(K,\mathfrak{p}_-)$ of Hermitian type.

Example 1. $(K,V)=(\mathbb{T}\times \mathrm{Sp}(n)\times \mathrm{SU}(2), \mathrm{Mat}(2n,2;\mathbb{C}))$, $n\geqslant 2$, $\mathrm{rank}\,(K,V)=3$. Fundamental highest weight vectors:

$$h_1 = z_{1,1}\,, \quad h_2 = z_{1,1}z_{2,2}-z_{2,1}z_{1,2}\,,$$
$$h_3 = \sum_{j=1}^{n}(z_{j,1}z_{2n-j+1,2}-z_{2n-j+1,1}z_{j,2})\,.$$

h_3 is an invariant for $\mathrm{Sp}(n)$. Put

$$q_1=\sum_{j,k}|z_{j,k}|^2\,, \quad q_2=\sum_{j<k}|z_{j,1}z_{k,2}-z_{k,1}z_{j,2}|^2\,, \quad q_3=|h_3(z)|^2,$$
$$D_1=\sum_{j,k} z_{j,k}\partial_{z_{j,k}}\,, \quad D_2=\sum_{j<k}(z_{j,1}z_{k,2}-z_{k,1}z_{j,2})(\partial_{z_{j,1}}\partial_{z_{k,2}}-\partial_{z_{k,1}}\partial_{z_{j,2}}),$$
$$D_3=h_3(z)h_3(\partial)\,.$$

Theorem 2 *For $\lambda=(l_1,l_2,l_3)$ we have*

$$p_\lambda(z,\overline{z})=\frac{1}{c_\lambda}\sum_{k=l_2}^{[(l_1+l_2)/2)]}(-1)^{k-l_2}\binom{l_1-k}{k-l_2}q_1^{l_1+l_2-2k}q_2^{k-l_2}\times$$
$$\times\sum_{j=l_3}^{l_2}(-1)^{j-l_3}\binom{l_2-l_3}{j-l_3}\frac{(l_1-l_3+1)^{\underline{j-l_3}}}{(l_1+l_2-2l_3+2n-2)^{\underline{j-l_3}}}\;q_2^{l_2-j}q_3^{j},$$

$$p_\lambda(z,\partial)=\frac{1}{c_\lambda}\sum_{k=l_2}^{[(l_1+l_2)/2]}(-1)^{k-l_2}\binom{l_1-k}{k-l_2}\prod_{a=2k}^{l_1+l_2-1}(D_1-a)\prod_{b=l_2}^{k-1}(D_2-bD_1+b^2-b)\times$$
$$\times\sum_{j=l_3}^{l_2}(-1)^{j-l_3}\binom{l_2-l_3}{j-l_3}\frac{(l_1-l_3+1)^{\underline{j-l_3}}}{(l_1+l_2-2l_3+2n-2)^{\underline{j-l_3}}}\times$$
$$\times\prod_{c=j}^{l_2-1}(D_2-cD_1+c^2-c)\prod_{d=0}^{j-1}(D_3-dD_1+d^2-(2n-1)d),$$

where $a^{\underline{l}}$ is a falling factorial power.

Example 2. $(K,V)=(\mathbb{T}^2\times \mathrm{SU}(n),\mathbb{C}^n\oplus\mathbb{C}^n)$, $n\geqslant 2$, $\mathrm{rank}\,(K,V)=3$. Fundamental highest weight vectors:

$$h_{1,1}=z_{1,1},\quad h_{1,2}=z_{1,2},\quad h_2=z_{1,1}z_{2,2}-z_{2,1}z_{1,2}.$$

Put

$$q_{1,1}=\sum_{j=1}^{n}|z_{j,1}|^2,\quad q_{1,2}=\sum_{j=1}^{n}|z_{j,2}|^2,\quad q_2=\sum_{j<k}|z_{j,1}z_{k,2}-z_{k,1}z_{j,2}|^2,$$
$$D_{1,1}=\sum_{j=1}^{n}z_{j,1}\partial_{z_{j,1}},\quad D_{1,2}=\sum_{j=1}^{n}z_{j,2}\partial_{z_{j,2}},$$
$$D_2=\sum_{j<k}(z_{j,1}z_{k,2}-z_{k,1}z_{j,2})(\partial_{z_{j,1}}\partial_{z_{k,2}}-\partial_{z_{k,1}}\partial_{z_{j,2}}).$$

Theorem 3 *For $\lambda=(l_{1,1},l_{1,2};l_2)$ we have*

$$p_\lambda(z,\overline{z})=\frac{1}{c_\lambda}\sum_j(-1)^{j-l_2}\binom{l_{1,1}+l_{1,2}-l_2-j}{j-l_2}\binom{l_{1,1}+l_{1,2}-2j}{l_{1,2}-j}q_{1,1}^{l_{1,1}-j}q_{1,2}^{l_{1,2}-j}q_2^j,$$

$$p_\lambda(z,\partial)=\frac{1}{c_\lambda}\sum_j(-1)^{j-l_2}\binom{l_{1,1}+l_{1,2}-l_2-j}{j-l_2}\binom{l_{1,1}+l_{1,2}-2j}{l_{1,2}-j}\times$$
$$\times\prod_{a=j}^{l_{1,1}-1}(D_{1,1}-a)\prod_{b=j}^{l_{1,2}-1}(D_{1,2}-b)\prod_{c=0}^{j-1}(D_2-c(D_{1,1}+D_{1,2})+c^2-c),$$

where $l_2\leqslant l\leqslant \min\{l_{1,1},l_{1,2}\}$.

Example 3. $(K,V)=(\mathbb{T}^2\times\mathrm{SU}(2)\times\mathrm{SU}(n),\mathbb{C}^2\oplus\mathrm{Mat}(2,n;\mathbb{C}))$, $n\geqslant 2$, $\mathrm{rank}\,(K,V)=4$. Fundamental highest weight vectors:

$$h_{1,0}=z_{1,0},\quad h_{1,1}=z_{1,1},\quad h_{2,0}=z_{1,0}z_{2,1}-z_{2,0}z_{1,1},\quad h_{2,1}=z_{1,1}z_{2,2}-z_{2,1}z_{1,2}.$$

Put

$$q_{1,0}=|z_{1,0}|^2+|z_{2,0}|^2,\qquad q_{1,1}=\sum_{j,k}|z_{j,k}|^2,$$
$$q_{2,0}=\sum_{j=1}^{n}|z_{1,0}z_{2,j}-z_{2,0}z_{1,j}|^2,\quad q_{2,1}=\sum_{j<k}|z_{1,j}z_{2,k}-z_{2,j}z_{1,k}|^2,$$
$$D_{1,0}=z_{1,0}\partial_{z_{1,0}}+z_{2,0}\partial_{z_{2,0}},\qquad D_{1,1}=\sum_{j,k}z_{j,k}\partial_{z_{j,k}},$$
$$D_{2,0}=\sum_{j<k}(z_{1,j}z_{2,k}-z_{2,j}z_{1,k})(\partial_{z_{1,j}}\partial_{z_{2,k}}-\partial_{z_{2,j}}\partial_{z_{1,k}}),$$
$$D_{2,1}=\sum_{j<k}(z_{1,j}z_{2,k}-z_{2,j}z_{1,k})(\partial_{z_{1,j}}\partial_{z_{2,k}}-\partial_{z_{2,j}}\partial_{z_{1,k}}).$$

Theorem 4 *For* $\lambda = (l_{1,0}, l_{1,1}, l_{2,0}, l_{2,1})$ *we have*

$$\begin{aligned} p_\lambda(z,\overline{z}) = {} & \frac{1}{c_\lambda} \sum_k \sum_j (-1)^{j-l_{2,0}} \times \\ & \times \binom{l_{1,0}+l_{1,1}-l_{2,0}+l_{2,1}-j}{j-l_{2,0}} \binom{l_{1,0}+l_{1,1}+l_{2,1}-2j}{l_{1,0}-j+k} \times \\ & \times \sum_l (-1)^{l-l_{2,1}} \binom{l_{2,0}-l_{2,1}}{l-l_{2,1}} \binom{j-l}{j-k} \frac{(l_{1,0}+l_{1,1}-l_{2,0}+1)^{\underline{l-l_{2,1}}}}{(l_{1,1}-l_{2,1})^{\underline{l-l_{2,1}}}} \times \\ & \times q_{1,0}^{l_{1,0}-j+k} q_{1,1}^{l_{1,1}+l_{2,1}-j-k} q_{2,0}^{j-k} q_{2,1}^{k}, \end{aligned}$$

$$\begin{aligned} p_\lambda(z,\partial) = {} & \frac{1}{c_\lambda} \sum_k \sum_j (-1)^{j-l_{2,0}} \times \\ & \times \binom{l_{1,0}+l_{1,1}-l_{2,0}+l_{2,1}-j}{j-l_{2,0}} \binom{l_{1,0}+l_{1,1}+l_{2,1}-2j}{l_{1,0}-j+k} \times \\ & \times \sum_l (-1)^{l-l_{2,1}} \binom{l_{2,0}-l_{2,1}}{l-l_{2,1}} \binom{j-l}{j-k} \frac{(l_{1,0}+l_{1,1}-l_{2,0}+1)^{\underline{l-l_{2,1}}}}{(l_{1,1}-l_{2,1})^{\underline{l-l_{2,1}}}} \times \\ & \times \prod_{a=j-k}^{l_{1,0}-1} (D_{1,0}-a) \prod_{b=j+k}^{l_{1,1}+l_{2,1}-1} (D_{1,1}-b) \times \\ & \times \sum_{s=0}^{j-k} (-1)^s \binom{j-k}{s} \prod_{t=k+s}^{j-1} (D_{2,0}-t(D_{1,0}+D_{1,1})+t^2-t) \times \\ & \times \prod_{c=0}^{k+s-1} (D_{2,1}-cD_{1,1}+c^2-c), \end{aligned}$$

where $l_{2,1} \leqslant k \leqslant [(l_{1,1}+l_{2,1})/2]$, $\max\{k, l_{2,0}\} \leqslant j \leqslant \min\{l_{1,0}+k, l_{1,1}+l_{2,1}-k\}$ *and* $l_{2,1} \leqslant l \leqslant \min\{l_{2,0}, l_{1,1}-l_{2,0}+l_{2,1}\}$.

Example 4. $(K,V) = (\mathbb{T}^2 \times \mathrm{Sp}(n), \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$, $n \geqslant 2$, $\mathrm{rank}\,(K,V) = 4$. Fundamental highest weight vectors:

$$h_{1,1} = z_{1,1}, \quad h_{1,2} = z_{1,2}, \quad h_2 = z_{1,1}z_{2,2} - z_{2,1}z_{1,2},$$
$$h_3 = \sum_{j=1}^{n} (z_{j,1}z_{2n-j+1,2} - z_{2n-j+1,1}z_{j,2}).$$

Put

$$
\begin{aligned}
q_{1,1} &= \sum_{j=1}^{2n} |z_{j,1}|^2, \quad q_{1,2} = \sum_{j=1}^{2n} |z_{j,2}|^2, \\
q_2 &= \sum_{j<k} |z_{j,1} z_{k,2} - z_{k,1} z_{j,2}|^2, \quad q_3 = |h_3(z)|^2, \\
D_{1,1} &= \sum_{j=1}^{2n} z_{j,1} \partial_{z_{j,1}}, \quad D_{1,2} = \sum_{j=1}^{2n} z_{j,2} \partial_{z_{j,2}}, \\
D_2 &= \sum_{j<k} (z_{j,1} z_{k,2} - z_{k,1} z_{j,2})(\partial_{z_{j,1}} \partial_{z_{k,2}} - \partial_{z_{k,1}} \partial_{z_{j,2}}), \quad D_3 = h_3(z) h_3(\partial).
\end{aligned}
$$

Theorem 5 *For $\lambda = (l_{1,1}, l_{1,2}; l_2, l_3)$ we have*

$$
\begin{aligned}
p_\lambda(z, \overline{z}) = \frac{1}{c_\lambda} \sum_k (-1)^{k-l_2} & \binom{l_{1,1} + l_{1,2} - l_2 - k}{k - l_2} \times \\
& \times \binom{l_{1,1} + l_{1,2} - 2k}{l_{1,2} - k} q_{1,1}^{l_{1,1}-k} q_{1,2}^{l_{1,2}-k} q_2^{k-l_2} \times \\
& \times \sum_j (-1)^{j-l_3} \binom{l_2 - l_3}{j - l_3} \frac{(l_{1,1} + l_{1,2} - l_2 - l_3 + 1)^{\underline{j-l_3}}}{(l_{1,1} + l_{1,2} - 2l_3 + 2n - 2)^{\underline{j-l_3}}} q_2^{l_2-j} q_3^j,
\end{aligned}
$$

$$
\begin{aligned}
p_\lambda(z, \partial) = \frac{1}{c_\lambda} \sum_k & (-1)^{k-l_2} \binom{l_{1,1} + l_{1,2} - l_2 - k}{k - l_2} \binom{l_{1,1} + l_{1,2} - 2k}{l_{1,2} - k} \times \\
& \times \prod_{a=k}^{l_{1,1}-1} (D_{1,1} - a) \prod_{b=k}^{l_{1,2}-1} (D_{1,2} - b) \prod_{c=l_2}^{k-1} (D_2 - c(D_{1,1} + D_{1,2}) + c^2 - c) \times \\
& \times \sum_j (-1)^{j-l_3} \binom{l_2 - l_3}{j - l_3} \frac{(l_{1,1} + l_{1,2} - l_2 - l_3 + 1)^{\underline{j-l_3}}}{(l_{1,1} + l_{1,2} - 2l_3 + 2n - 2)^{\underline{j-l_3}}} \times \\
& \times \prod_{d=j}^{l_2-1} (D_2 - d(D_{1,1} + D_{1,2}) + d^2 - d) \times \\
& \times \prod_{e=0}^{j-1} (D_3 - e(D_{1,1} + D_{1,2}) + e^2 - (2n-1)e),
\end{aligned}
$$

where $l_2 \leqslant k \leqslant \min\{l_{1,1}, l_{1,2}\}$, $l_3 \leqslant l \leqslant l_2$.