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Whittaker functions with one-dimensional K-type on a semisimple Lie group of Hermitian type

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We give an explicit formula for the Whittaker function with one-dimensional K-type on a simple Lie group of Hermitian type.

Keywords: semisimple Lie groups, Hermitian symmetric spaces, K-types, Harish-Chandra c-function, Whittaker functions

Introduction

The radial part of the zonal spherical function $\phi_\lambda(a)$ on a Riemannian symmetric space can be written in the form

$$\phi_\lambda(a) = \sum_{w \in W} c(w\lambda)\Phi_{w\lambda}(a) \tag{1}$$

for generic λ, where W is the Weyl group, $c(\lambda)$ is the Harish-Chandra c-function, and $\Phi_\lambda(a)$ is the joint eigenfunction of the radial parts of the invariant differential operators given by a series expansion $\Phi_\lambda(a) \sim a^{\lambda-\rho}$ as $a \to \infty$. The coefficients of the series expansion are determined recursively from the equation for the radial part of the Casimir operator.

Hashizume [2] proved a formula that is similar to (1) for the class one Whittaker function given by the Jacquet integral on a real semisimple Lie group. In this article, we generalize the result of Hashizume for the Whittaker function with one-dimensional K-type on a simple Lie group of Hermitian type. We also give the formula for the radial part of the Casimir operator for the Siegel-Whittaker function on a Hermitian symmetric space of tube type.

We will give the proof in a forthcoming paper.

§ 1. Jacquet–Whittaker functions

Let G/K be an irreducible Hermitian symmetric space with G a simple Lie group of Hermitian type and K a maximal compact subgroup of G. Let $G = NAK$ be a Iwasawa decompostion. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$ denote the Lie algebras of G, N, A, K respectively. Fix an inner product $\langle\,,\,\rangle$ on $\mathfrak{a}$.

Let Σ denote the restricted root system for G/K, and Σ^+ the positive system corresponding to N. Let $\Psi \subset \Sigma^+$ denote the set of simple roots in Σ^+. Let m_α be the multiplicity of $\alpha \in \Sigma$ and $\rho = (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. Let W be the Weyl group for Σ and $s_\alpha \in W$ denote the simple reflection corresponding to $\alpha \in \Psi$. Let $\ell(w)$ denote the length of $w \in W$, $w_0 \in W$ the longest element of W, and $\overline{w}_0 \in N_K(\mathfrak{a})$ a representative of w_0.

Let η be a normalized nondegenerate unitary character of N and χ_ℓ a unitary character of K. Here $\ell \in \mathbb{Z}$ if G is a real form of the simply connected complex Lie group with the Lie algebra $\mathfrak{g}_\mathbb{C}$, and $\ell \in \mathbb{R}$ if G is simply connected. Let $\lambda \in \mathfrak{a}^*_\mathbb{C}$. Define the function $1_{\lambda,\ell}$ on $G = NAK$ by

$$1_{\lambda,\ell}(nak) = a^{\lambda+\rho}\chi_{-\ell}(k), \quad n \in N,\, a \in A,\, k \in K.$$

Define the Jacquet-Whittaker function with one-dimensional K-type by

$$W_\ell(\lambda,\eta;g) = \int_N \eta(n)^{-1}\, 1_{\lambda,\ell}(\overline{w}_0^{\,-1} ng)\, dn,$$

where dn is a suitably normalized invariant measure on N (cf. [4], [7], [2], [1]).

The following properties characterize the meromorphic continuation in λ of $u(a) = W_\ell(\lambda,\eta;a)$:

(i) $u(nak) = \eta(n)\, u(a)\, \chi_{-\ell}(k), \quad n \in N,\, a \in A,\, k \in K$;

(ii) $D\,u = \gamma_\ell(D)(\lambda)u, \quad \forall D \in \mathbb{D}_\ell(G/K)$, where $\mathbb{D}_\ell(G/K)$ is the algebra of invariant differential operators on the homogeneous line bundle over G/K associated with χ_ℓ and $\gamma_\ell : \mathbb{D}_\ell(G/K) \to S(\mathfrak{a})^W$ the Harish-Chandra isomorphism;

(iii) $u(a)$ is of moderate growth, i.e. $|u(a)| \leqslant C\exp\{k|\log a|\}$, $C > 0$, $k > 0$;

(iv) $u(a) \sim \chi_\ell(\overline{w}_0)\, c_\ell(\lambda)\, a^{w_0\lambda+\rho}$ as $a \to \infty$.

For the root system of type BC_r, there are three root length, say long, middle, and short. If G/K is of tube type, then the root system is of type C_r and there are no short roots. The Harish-Chandra c-function for the one-dimensional K-type χ_ℓ is given by

$$c_\ell(\lambda) = c_1 \prod_{\alpha\in\Sigma^+,\ \text{middle}} \frac{2^{-\lambda_\alpha}\Gamma(\lambda_\alpha)}{\Gamma\left(\frac{1}{2}(\lambda_\alpha + \frac{1}{2}m_\alpha + 1)\right)\Gamma\left(\frac{1}{2}(\lambda_\alpha + \frac{1}{2}m_\alpha)\right)} \times$$

$$\times \prod_{\alpha\in\Sigma^+,\ \text{long}} \frac{2^{-\lambda_{\alpha/2}}\Gamma(\lambda_{\alpha/2})}{\Gamma\left(\frac{1}{2}(\lambda_{\alpha/2} + \frac{1}{2}m_{\alpha/2} + 1 + \ell)\right)\Gamma\left(\frac{1}{2}(\lambda_{\alpha/2} + \frac{1}{2}m_{\alpha/2} + 1 - \ell)\right)},$$

where $\lambda_\alpha = \dfrac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ and c_1 is determined by $c_0(\rho) = 1$ (cf. [8], [9]).

The set of the simple roots is given by

$$\Psi = \{e_r - e_{r-1}, \ldots, e_2 - e_1, 2e_1\} \quad \text{(tube type)},$$
$$\Psi = \{e_r - e_{r-1}, \ldots, e_2 - e_1, e_1\} \quad \text{(non-tube type)},$$

where r is the rank of the symmetric space. There are three root length, long ($2e_i$), middle ($e_i \pm e_j$), and short (e_i).

Define numbers $M_\ell(\lambda, \eta; w)$, $w \in W$, recursively by

$$M_\ell(\lambda, \eta; e) = 1,$$
$$M_\ell(\lambda, \eta; w\, s_\alpha) = M_\ell(\lambda, \eta; w)\, M_\ell(w\lambda, \eta; s_\alpha), \quad \ell(w\, s_\alpha) = \ell(w) + 1.$$

Then $M_\ell(\lambda, \eta; s_\alpha)$, $\alpha \in \Psi$, are given according to the length of α as follows (cf. [4], [7], [1], [2]):

(i) If $\alpha \in \Psi$ is a middle root, then

$$M_\ell(\lambda, \eta; s_\alpha) = 2^{-4\lambda_\alpha}\, e_\alpha(\lambda)\, e_\alpha(-\lambda)^{-1},$$
$$e_\alpha(\lambda)^{-1} = \Gamma\left(\frac{1}{2}(\lambda_\alpha + m_\alpha + 1)\right) \Gamma\left(\frac{1}{2}(\lambda_\alpha + m_\alpha)\right).$$

(ii) If $\alpha = e_1 \in \Psi$ is a short root (the case of non-tube type), then

$$M_\ell(\lambda, \eta; s_\alpha) = 2^{-4\lambda_\alpha}\, e_\alpha(\lambda, \ell)\, e_\alpha(-\lambda, \ell)^{-1},$$
$$e_\alpha(\lambda, \ell)^{-1} = \Gamma\left(\frac{1}{2}(\lambda_\alpha + \frac{1}{2}m_\alpha + 1 + \ell)\right) \Gamma\left(\frac{1}{2}(\lambda_\alpha + \frac{1}{2}m_\alpha + 1 - \ell)\right).$$

(iii) If $\alpha = 2e_1 \in \Psi$ is a long root (the case of tube type), then

$$M_\ell(\lambda, \eta; s_\alpha) = 2^{-2\lambda_\alpha}\, \frac{\Gamma\left(-\lambda_\alpha + \frac{1}{2} - \frac{\ell}{2}\right)}{\Gamma\left(\lambda_\alpha + \frac{1}{2} - \frac{\ell}{2}\right)}.$$

Then the Whittaker function $W_\ell(\lambda, \eta; g)$ satisfies the functional equation

$$W_\ell(\lambda, \eta; g) = M_\ell(\lambda, \eta; w)\, W_\ell(w\lambda, \eta; g), \quad w \in W.$$

Let $\Delta(\Omega)$ denote the radial part of the Casimir operator Ω with respect to the Iwasawa decomposition $G = NAK$ for representations η and χ_ℓ from the left and right respectively. Then $e^{-\rho} \circ (\Delta(\Omega) + \langle \rho, \rho \rangle) \circ e^{\rho}$ is given by

$$\Omega_{\mathfrak{a}} - \sum_{\alpha \in \Psi} \langle \alpha, \alpha \rangle\, e^{2\alpha} - 4\langle e_1, e_1 \rangle\, \ell\, e^{2e_1} \quad \text{(tube type)},$$
$$\Omega_{\mathfrak{a}} - \sum_{\alpha \in \Psi} \langle \alpha, \alpha \rangle\, e^{2\alpha} \quad \text{(non-tube type)},$$

where $\Omega_{\mathfrak{a}}$ is the Casimir operator on $\mathfrak{a}$. The operator is the same as the class one case for non-tube type, and there is an additional term for tube type:

$$e^{-\rho} \circ (\Delta(\Omega) + \langle \rho, \rho \rangle) \circ e^{\rho} = \sum_{i=1}^{r} \frac{\partial^2}{\partial t_i^2} - \frac{1}{2} \sum_{i=1}^{r-1} e^{t_{i+1}-t_i} - e^{2t_1} - \ell\, e^{t_1}. \qquad (2)$$

Radial parts of invariant differential operators give a family of commuting differential operators, which prove complete integrability of Toda model with the above Schrödinger operator.

Let $\Phi_\lambda(a)$ denote the series solution of

$$(e^{-\rho} \circ (\Delta(\Omega) + \langle \rho, \rho \rangle) \circ e^{\rho})\Phi_\lambda = \langle \lambda, \lambda \rangle \Phi_\lambda$$

of the form

$$\Phi_\lambda(a) = a^\lambda \sum_{\mu \in \Lambda} b_\mu(\lambda) a^\mu, \quad b_0(\lambda) = 1,$$

where Λ denotes the family of linear combinations $\sum n_\alpha \alpha$ with $\alpha \in \Psi$ and $n_\alpha \in \mathbb{Z}^+$. Then $e^\rho \Phi_\lambda$ is an eigenfunction of $\Delta(D)$ for all $D \in \mathbb{D}_l(G/K)$, and functions $\Phi_{w\lambda}$, $w \in W$, form a basis of the space of the joint eigenfunctions for generic λ.

Now we state the main result of this article.

Theorem 1 *For generic λ we have*

$$W_\ell(\lambda, \eta\,;\, a) = \sum_{w \in W} M_\ell(\lambda, \eta\,;\, w_0 w)\, c_\ell(w_0 w \lambda)\, \Phi(w\lambda, \eta\,;\, a).$$

If $\ell = 0$, the above theorem is a special case of the result of Hashizume [2].

As in the case of the Whittaker functions on semisimple Lie groups of rank one, Whittaker functions depend "essentially" on reduced root system consisting of inmultiplicative roots in Σ and λ (and ℓ for tube type). We define the Whittaker function associated with C_r-type root system. For $G = \mathrm{Sp}(r, \mathbb{R})$, the system Σ is of type C_r and $m_\alpha = 1$ for all $\alpha \in \Sigma$. Define a function $\mathcal{W}_{\lambda,\ell}(a)$ on A by

$$\mathcal{W}_{\lambda,\ell}(a) = d_\ell(\lambda)^{-1} a^{-\rho}\, W_\ell(\lambda, \eta\,;\, a)$$

where

$$\begin{aligned} d_\ell(\lambda) &= \chi_\ell(\overline{w}_0)\, c_\ell(\lambda) \prod_{\alpha \in \Sigma^+,\, \mathrm{middle}} 2^{-\lambda_\alpha}\, \Gamma(\lambda_\alpha)^{-1} \times \\ &\times \prod_{\alpha \in \Sigma^+,\, \mathrm{long}} 2^{\lambda_\alpha}\, \Gamma\left(\lambda_\alpha + \frac{1}{2} + \frac{\ell}{2}\right) \Gamma(2\lambda_\alpha)^{-1} \end{aligned} \qquad (3)$$

with $c_\ell(\lambda)$ the c-function for $\mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(r)$. Then $\mathcal{W}_{\lambda,\ell}(a)$ is W-invariant with respect to λ and

$$\lim_{a \to \infty} a^{-w_0\lambda}\, \mathcal{W}_{\lambda,\ell}(a) = d_\ell(\lambda)^{-1} c_\ell(\lambda).$$

Corollary 1 *Let G/K be an irreducible Hermitian symmetric space of tube type. Then we have*

$$W_\ell(\lambda, \eta\,;\, a) = d_\ell(\lambda) a^\rho \mathcal{W}(\lambda, \ell\,;\, a),$$

where $d_\ell(\lambda)$ is given by (3) *with $c_\ell(\lambda)$ the c-function for G/K.*

We can regard $\mathcal{W}_{\lambda,\ell}(a)$ as a multivariable analogue of the classical Whittaker function $W_{\varkappa,\mu}(z)$, which is associated to the root system of type C_r. In a similar way, we can define a multivariable analogues of the modified Bessel function of the second kind associated with reduced root systems from class one Whittaker functions on real split simple Lie groups.

§ 2. Siegel-Whittaker functions

Let G/K be an irreducible Hermitian symmetric space of tube type, $P_s = L_s \ltimes N_s$ Siegel parabolic subgroup of G, $R = (L_s \cap K) \ltimes N_s$. Then we have the generalized Cartan decomposition $G = RAK$. Let η be a normalized unitary character of N_s that is fixed by $L_s \cap K$. We consider functions on G that satisfy

$$u(rak) = \mathrm{triv}_{L_s \cap K} \cdot \eta(r)^{-1}\, u(a)\, \chi_{-\ell}(k), \quad r \in R,\, a \in A,\, k \in K.$$

Then the radial part $\Delta(\Omega)$ of the Casimir operator Ω for such u satisfies

$$\begin{aligned}&\delta^{1/2} \circ (\Delta(\Omega) + \langle \rho, \rho \rangle) \circ \delta^{-1/2} \\ &= \sum_{i=1}^r \frac{\partial^2}{\partial t_i^2} + \sum_{1 \leqslant i < j \leqslant r} \frac{m_{e_j - e_i}(2 - m_{e_j - e_i})}{2 \sinh^2(t_j - t_i)} - 4 \sum_{i=1}^r (e^{4t_i} + \ell e^{2t_i}),\end{aligned} \tag{4}$$

where

$$\delta = \prod_{\alpha \in \Sigma^+, \mathrm{middle}} (\sinh \alpha)^{m_\alpha} \prod_{i=1}^r e^{-t_i/2}.$$

Radial parts of invariant differential operators give a commuting family of differential operators containing the above Schrödinger operator, which prove integrability of the model.

There exists a unique joint eigenfunction globally defined on A and of moderate growth up to constant multiples ([11]). This function was studied by Ishii [3] for $G = \mathrm{SO}_0(2, n)$.

Remark 1 (i) It is known that the Schrödinger operators (2) and (4) give quantum integrable models (cf. [5]). Radial parts of invariant differential operators give a family of commuting differential operator, which prove complete integrability of the models with the Schrödinger operators (2) and (4). These group theoretic interpretations seem to be new.

(ii) We can prove in a similar way as [10] that the C-type Whittaker function $\mathcal{W}_{\lambda,\ell}(a)$ is a degenerate limit of the BC-type Heckman-Opdam hypergeometric function. The Siegel-Whittaker functions of moderate growth is also a degenerate limits of the Heckman-Opdam hypergeometric functions (cf. [6]).

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