Unrefinable, simple graded Lie-algebras

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Abstract

We deal with the problem of the classification of simple graded Lie algebras. By a graded Lie algebra we mean a Lie algebra equipped with its grading by an arbitrary Abelian grading group H. In 1969 V.Kac classified finite-dimensional semisimple Lie algebras graded by a cyclic group. We study the opposite case: Lie algebras graded by a maximal possible grading group H, or equivalently, graded Lie algebras which grading subspaces are as small, as possible. We call such algebra the unrefinable graded Lie algebra. The result of this work is the classification of finite-dimensional, unrefinable, simple graded Lie algebras over the field of complex numbers (it is not yet completed for Lie algebras E_7 and E_8). This result is equivalent to the classification of maximal, commutative, diagonalizable subgroups of groups of automorphisms of semisimple Lie algebras.

1 Introduction

In this work we have classified the important class of finite-dimensional, simple graded Lie algebras over the field of complex numbers. This class consists of unrefinable graded Lie algebras (see definition 3).

Let us first clarify the terminology. "A graded algebra" is an algebra with a fixed grading of it (see definition 1). Graded Lie algebras form the category, in which a morphism is defined as a morphism of Lie algebras compatible with gradings. A simple object in this category is called a simple graded Lie algebra. A grading of a simple Lie algebra is the simple graded Lie algebra; converse statement is not true.

A grading of the algebra is called the refinement of another grading iff any grading subspace of the first grading is contained in a certain grading subspace of the second grading. A grading of an algebra is called unrefinable iff there is no refinements of it. An algebra with an unrefinable grading is called an unrefinable graded algebra (see definition 3).

Gradings of Lie algebras are widely used in the Lie group theory and in other theories where Lie algebras are needed. There is a simple explanation of this fact: for an algebra in a basis compatible with a grading, the structure constants are much "simpler" than in a basis in general position. For instance, in the first case, a lot of the structure constants are equal to zero. Thus, the classification of graded Lie algebras is important for finding different kinds of "simple" bases of an algebra.

Surprisingly, this quite natural and important classification problem was not solved even for semisimple Lie algebras over the field of complex numbers, although many of such algebras are well-known, even became classical objects.

Let us show why it is natural to restrict the classification to the unrefinable gradings.

First, a grading with "smallest" grading subspaces allows to choose a "simplest" basis. We illustrate this "simplicity" by a grading with only one-dimensional grading subspaces. Let $\{E_i\}$ be a basis in general position of an algebra $\mathfrak{g}$. Then $[E_i, E_j] = C(i,j,k)E_k$ and the structure constants $C(i,j,k)$ depend on three variables: i, j, k. Assume, there exists a grading of $\mathfrak{g}$ with one-dimensional grading subspaces; denote by H the grading group. Consider a basis $\{E_a|\ a \in H\}$ which is compatible with the grading. Then $[E_a, E_b] = f(a,b)E_{a+b}$ $(a, b \in H)$. Therefore, $C(a,b,c) = 0$ if $a+b \neq c$ and $C(a,b,a+b) = f(a,b)$. Thus, the structure constants essentially depend only on two variables. Therefore, this case (of one-dimensional grading subspaces) leads to the very simple basis of an algebra.

In general, grading subspaces of an unrefinable grading can be of higher dimension. Nevertheless, the example shows why the smallest grading subspaces leads to the simplest basis. Moreover, an arbitrary grading of an algebra can be refined to an unrefinable grading. Therefore, a basis compatible with the

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unrefinable grading is compatible also with the first one. Thus, if we are interesting in "simple" bases of an algebra, we can restrict the classification problem to the case of unrefinable gradings.

Second, the classification of unrefinable gradings is an essential step in the classification of all gradings of Lie algebras. Indeed, any grading of an algebra can be obtained from an unrefinable grading by the procedure, inverse to the refinement. The essence of this inverse procedure is gluing together several grading subspaces to obtain grading subspaces of a new grading. This procedure can be defined purely in the terms of the invariants of the unrefinable grading, which are combinatorial objects. Thus, the complete classification of gradings can be derived from the classification of unrefinable gradings.

The famous example of an unrefinable grading is the Cartan root decomposition of a semisimple Lie algebra. It is not a unique example: there exist other unrefinable gradings of semisimple Lie algebras. The list of them includes among others:

- the gradings which lie in the base of Clifford algebras and generalized Clifford algebras introduced by Morinaga and Nono [10];

- the gradings of Popovichi (see [11]);

- the gradings by Jordan subgroups [2], [3] (not all of them are unrefinable gradings but the majority are);

- the gradings of Hesselink [7];

- two gradings of E_8 by the grading groups $H_1 = \mathbb{Z}_2^8$ and $H_2 = \mathbb{Z}_2^9$. We do not know a reference for the explicit description of these two interesting gradings. Their existence is the trivial corollary of the work of Adams [1] in which maximal commutative subgroups $\mathbb{Z}_2^8$ and $\mathbb{Z}_2^9$ of the Lie group E_8 were found. These two gradings were also discovered independently by the author (but were not published): they are the refinements of the nice grading of E_8 by the Jordan subgroup $H = \mathbb{Z}_2^5$ (see [2], [3]). The last grading was rediscovered independently by Tompson and was used in the construction of the special integer lattice in E_8 [13], [12]).

It is not easy to list all known unrefinable grading: sometimes they appeared in publications implicitly, as a technical tool. For example, in the paper of Bernstein, Gelfand, Gelfand [4] special gradings (which can be called "the Cartan gradings mod 2") were used in the construction of models of representations of compact Lie groups. However, in this paper these gradings were not even defined as an independent object.

One can also find the use of unrefinable gradings in the construction of some quantum groups, in which the basis compatible with the structure of the generalized Clifford algebra is used.

Unrefinable gradings of simple Lie algebras were also essentially used in the papers of Kostrikin and his co-authors, in which the orthogonal Cartan decompositions were studied (see [9]).

Let us shortly describe the result of the work, i.e. the classification of finite-dimensional, unrefinable, simple graded Lie algebras over the field of complex number.

There are four series of unrefinable gradings of simple Lie algebras. Cartan root decompositions are particular cases of these series with special values of parameters. Graded Lie algebras from these series we call classical graded Lie algebras (all of them are gradings of the classical Lie algebras). These four series are more or less known objects. Our result is the computing their invariants and the presenting them in the frame of general theory (section 5).

The main invariants of an unrefinable graded Lie algebra are:

- the generalized root system (shortly, root system);

- the group of diagonal automorphisms of the graded Lie algebra, which is a diagonalizable subgroup of the group of all automorphisms of the Lie algebra;

- the generalized Weyl group.

In section 2 exact definitions of these invariants are given and their properties are stated.

One of the series is constructed from graded associative algebras. In the section 3 the classification of finite-dimensional, unrefinable, simple graded associative algebras is given.

In addition to the four series, there are several exceptional unrefinable gradings of simple Lie algebras. All of them are gradings of the simple Lie algebras $D_4, G_2, F_4, E_6, E_7, E_8$. There are: 2 unrefinable gradings of $\mathfrak{g} = G_2$; 3 special unrefinable gradings of $\mathfrak{g} = D_4$ (plus gradings, which are included into the series); 4 unrefinable gradings of $\mathfrak{g} = F_4$; 12 unrefinable gradings of $\mathfrak{g} = E_6$; 12? unrefinable gradings of $\mathfrak{g} = E_7$; < 20 unrefinable gradings of $\mathfrak{g} = E_8$.

In the section 6 are listed all exceptional unrefinable gradings of Lie algebras D_4, G_2, F_4, E_6. We plan to present the complete list of exceptional unrefinable gradings of E_7 and E_8 together with the missing proofs in a separate publication.

We prove that the gradings listed above are all unrefinable gradings of the simple Lie algebras. In addition, there exist unrefinable, simple graded Lie algebras, which are not gradings of simple Lie algebras. All of them are gradings of semisimple Lie algebras. Moreover, all of them can be constructed from the gradings of simple Lie algebras by the use of the construction, which we call "an induced graded algebra". This construction is the direct generalization of the construction used by V.Kac for the classification of cyclic gradings [8]. We prove that all finite-dimensional, unrefinable, simple graded Lie algebras are induced from those unrefinable graded Lie algebras, which are gradings of simple Lie algebras (section 4).

The present paper is an attempt to describe in details the common technique for dealing with unrefinable gradings of Lie algebras. On the other hand, we omit some of the proofs which are rather standard or well-known. We want to point out that the technique, developed in this work, can be used for studying infinite-dimensional graded Lie algebras.

2 Definitions and denotations

Definition 1 *Let $\Gamma = (\mathfrak{g}, \Omega)$ be a pair, which consists of a Lie algebra $\mathfrak{g}$ and a set $\Omega = \{V \mid V \subset \mathfrak{g}\}$ of linear subspaces V of $\mathfrak{g}$. Then Γ is called the graded Lie algebra iff:*
(1) $\mathfrak{g} = \sum^{\bullet}_{V\in\Omega} V$;
(2) if $V_1, V_2 \in \Omega$ then $[V_1, V_2] = 0$ or $[V_1, V_2] \subset V_3$ for a certain $V_3 \in \Omega$;
(3) there exists an embedding ρ of Ω into an Abelian group H, such that if $[V_1, V_2] \neq 0$ and $[V_1, V_2] \subset V_3$ then $\rho(V_1) + \rho(V_2) = \rho(V_3)$.

A subspace $V \in \Omega$ is called a root subspace, or a grading subspace.

A morphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ of graded Lie algebras Γ_1, Γ_2 is defined as a morphism of Lie algebras $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that the image of any root subspace of Γ_1 is contained in a root subspace of Γ_2.

Let a graded Lie algebra $\Gamma = (\mathfrak{g}, \Omega)$ be given. Then we call $\mathfrak{g}$ the underlying Lie algebra. Conversely, a structure of a graded Lie algebra on a given Lie algebra $\mathfrak{g}$ is called a grading of $\mathfrak{g}$.

In Section 3 we deal also with graded associative algebras. The definition is just analogous to the definition 1. Particularly, we require that the group H is an Abelian group. Graded associative algebras with non-Abelian H also can be defined, but they are not needed in this work.

Definition 2 *Let graded Lie algebra $\Gamma = (\mathfrak{g}, \Omega)$ be given. Define an Abelian group H_{gr}. Its generators are elements $V \in \Omega$ and defining relations are: $V_1 + V_2 = V_3$ for any triple $V_1, V_2, V_3 \in \Omega$ such that $[V_1, V_2] \subset V_3$ and $[V_1, V_2] \neq 0$. We call H_{gr} the grading group of Γ. The image of the tautological map $\rho : \Omega \rightarrow H_{gr}$ we call generalized root system of Γ, or shortly root system. Its elements we call roots.*

It is easy to check that the tautological map of Ω into H_{gr} is an embedding. Therefore, an Abelian group H in the definition 1 can be chosen canonically: $H = H_{gr}$.

A grading subspace is in the same time a generator of the grading group. In order to distinguish these two roles we use denotations: $V \in \Omega$ denotes a root subspace in a Lie algebra $\mathfrak{g}$; $\rho(V) \in \rho(\Omega)$ denotes a root, i.e. an element of grading group H_{gr}. Conversely, for a root $\alpha \in \rho(\Omega)$ a relevant root subspace is denoted by V_α.

Let us define the main invariants of a graded Lie algebra $\Gamma = (\mathfrak{g}, \Omega)$.

First invariant is the generalized root system $\rho(\Omega) \subset H_{gr}$. In the case of Cartan root decomposition the generalized root system $\rho(\Omega)$ coincides with the classical root system enlarged by zero.

Denote by G the group $\mathrm{Aut}\,\mathfrak{g}$ of all automorphisms of underlying the Lie algebra $\mathfrak{g}$; G is a linear algebraic group. The automorphism group $\mathrm{Aut}\,\Gamma = \{g \in G \mid g(V) \in \Omega\ \forall V \in \Omega\}$ of the graded Lie algebra Γ is a closed subgroup of G. The group $A(\Gamma) = \{g \in \mathrm{Aut}\,\Gamma \mid g|_V = \lambda_V E, \lambda_V \in \mathbb{C}^*, \forall V \in \Omega\}$ (E denotes the unit matrix) we call the group of diagonal automorphisms of Γ. Clearly, $A(\Gamma)$ is a diagonalizable [1] subgroup of G.

Second invariant of Γ is the pair $(G, A(\Gamma))$. In the case of Cartan root decomposition $A(\Gamma)$ is the maximal torus of G.

The group $\mathrm{Aut}\,\Gamma$ acts naturally by automorphisms of the grading group H_{gr}. Third invariant is the image of $\mathrm{Aut}\,\Gamma$ in $\mathrm{Aut}\,H_{gr}$. We denote this image by $\widehat{W}$. Thus, $\widehat{W}$ is a subgroup of the automorphism

[1] A subgroup A of G is called diagonalizable if in a representation of G its image is a subgroup of the group of all diagonal matrices.

group of an Abelian group. In the case of Cartan root decomposition group $\widehat{W}$ coincides with the extension of Weyl group with the group of automorphisms of Dynkin diagram.

Lemma 1 *The group $A(\Gamma)$ is isomorphic to the group of characters $H^{\#}_{gr}$ of the grading group H_{gr}.*

Proof is standard. □

Definition 3 *A morphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ of graded Lie algebras is called an refinement iff it is an isomorphism of underlying Lie algebras.*

Γ_1 is called a refinement of Γ_2 iff there exists a refinement $\phi : \Gamma_1 \rightarrow \Gamma_2$.

Γ is called unrefinable graded Lie algebra iff there is no non-trivial (i.e. which are not isomorphisms of graded Lie algebras) refinements of Γ.

The definition can be reformulated as follows: morphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ is an refinement iff for every $V \in \Omega_2$ there exist $W_1, \ldots, W_k \in \Omega_1$ such that $V = \phi(W_1) + \phi(W_2) + \ldots + \phi(W_k)$. Thus, a graded Lie algebra $\Gamma = (\mathfrak{g}, \Omega)$ is an unrefinable one iff there is no gradings of underlying Lie algebra $\mathfrak{g}$, such that any root subspace of it is contained in a certain root subspace of Γ.

Gradings of Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with a special class of diagonalizable subgroups of the group $G = \operatorname{Aut}\mathfrak{g}$. For shortness, we formulate this standard fact only for unrefinable gradings.

Proposition 1 .
(1) $\Gamma = (\mathfrak{g}, \Omega)$ *is an unrefinable graded Lie algebra iff $A(\Gamma)$ is a maximal diagonalizable subgroup of the group $G = \operatorname{Aut}\mathfrak{g}$;*
(2) *every maximal diagonalizable subgroup A of G coincides with the group $A(\Gamma)$ for an unrefinable grading Γ of $\mathfrak{g}$;*
(3) $\Gamma_1 = (\mathfrak{g}_1, \Omega_1)$ *is isomorphic to* $\Gamma_2 = (\mathfrak{g}_2, \Omega_2)$ *iff there exists an isomorphism $\phi : G_1 \rightarrow G_2$ such that* $\phi(A(\Gamma_1)) = A(\Gamma_2)$

Thus, the problem of classification of unrefinable graded Lie algebras is equivalent to the problem of classification of maximal diagonalizable subgroups of the groups of automorphisms of Lie algebras.

The invariants defined above for a graded Lie algebra can be defined also for a graded associative algebra. The analog of proposition 1 also is true. We omit the precise formulations because they are evident.

3 Unrefinable simple graded associative algebras

Two series of unrefinable associative algebras are well-known. They are the generalized Clifford algebras and the Cartan grading of the matrix algebras.

3.1 Generalized Clifford algebras

Let H be an Abelian group and $\xi(a, b)$ be a 2-cocycle on H with values in $\mathbb{C}^*$. Denote by $\mathbb{C}[H]$ the group algebra of H. We define the new multiplication on $\mathbb{C}[H]$ by formula: $x * y = \xi(x, y)xy$ for any $x, y \in H$. Denote this new algebra by $\mathbb{C}[H]_\xi$ and denote by $\mathcal{H}$ the set of all one-dimensional subspaces generated by elements of the group H.

Proposition 2 *The pair $\Sigma_{H,\xi} = (\mathbb{C}[H]_\xi, \mathcal{H})$ is a graded associative algebra. It is unrefinable, simple graded algebra. $\Sigma_{H,\xi}$ is isomorphic to $\Sigma_{H',\xi'}$ iff there exists an isomorphism $\phi : H \rightarrow H'$ such that cocycle $\phi^*\xi'$ is cohomological to ξ.*

Proof. The condition of the associativity of a multiplication "$*$" is just equivalent to the condition that ξ is 2-cocycle. $\Sigma_{H,\xi}$ is unrefinable graded algebra because all its root subspaces are one-dimensional. It is easy to check the simplicity of $\Sigma_{H,\xi}$ and to verify the second statement. □

Thus, the graded algebra $\Sigma_{H,\xi}$ is uniquely defined by the orbit of cohomology class $[\xi] \in \mathrm{H}^2(H, \mathbb{C}^*)$ under the action of the group $\operatorname{Aut} H$. The graded algebra $\Sigma_{H,\xi}$ is called the generalized Clifford algebra [10]. Evidently, it is finite-dimensional iff a group H is finite.

3.2 The Cartan grading of a matrix algebra

Denote by $E_{i,j}$ an elementary matrix of the order $n \times n$. Denote by $V_{i,j}$ the one-dimensional subspace generated by $E_{i,j}$, $i \neq j$, by V_0 the subspace of all diagonal matrices and by Ω the set $\{V_0, V_{i,j} \mid i,j = 1,\dots,n,\ i \neq j\}$.

Proposition 3 *The pair* $\mathrm{M}_n^c = (\mathrm{M}_n, \Omega)$, *which consists of the matrix algebra* M_n *and the set of subspaces* Ω *is a graded associative algebra. It is unrefinable simple graded algebra.*

Proof is evident. □

3.3 Classification theorem

The tensor product of two graded associative algebras is defined as follows:

$$\Sigma_1 \otimes \Sigma_2 = (S_1 \otimes S_2, \Omega_1 \otimes \Omega_2) \quad \text{where } \Omega_1 \otimes \Omega_2 = \{V_1 \otimes V_2 \mid V_1 \in \Omega_1, V_2 \in \Omega_2\}$$

Evidently, $\Sigma_1 \otimes \Sigma_2$ is a graded associative algebra.

Let us compute the tensor products of the graded algebras, defined above.

For cocycles $\xi_1 \in \mathrm{Z}^2(H_1, \mathbb{C}^*)$ and $\xi_2 \in \mathrm{Z}^2(H_2, \mathbb{C}^*)$ denote by $\xi_1 \cdot \xi_2$ the 2-cocycle on the group $H_1 \oplus H_2$; it is defined by the formula: $(\xi_1 \cdot \xi_2)(x_1 \oplus y_1, x_2 \oplus y_2) = \xi_1(x_1, x_2) \cdot \xi_2(y_1, y_2)$.

Lemma 2 .
(1) $\Sigma_{H_1,\xi_1} \otimes \Sigma_{H_2,\xi_2} = \Sigma_{H_1 \oplus H_2, \xi_1 \cdot \xi_2}$;
(2) *the graded algebra* $\mathrm{M}_m^c \otimes \mathrm{M}_n^c$ *is refinable graded algebra if* $n, m > 1$;
(3) *the algebra* $\Sigma_{H,\xi} \otimes \mathrm{M}_n^c$ *is an unrefinable, simple graded algebra.*

Proof is trivial. □

Theorem 1 *Let* Σ *be a finite-dimensional, unrefinable, simple graded associative algebra. Then* Σ *is isomorphic to an algebra* $\Sigma_{H,\xi} \otimes \mathrm{M}_n^c$. *(Partial cases:* $H = \{0\}$ *and* $\Sigma = \mathrm{M}_n^c$; $n = 1$ *and* $\Sigma = \Sigma_{H,\xi}$ *are included.)*

The sketch of the proof is given in the further subsections.

3.4 The induced graded algebra

Let us describe the construction of a new graded algebra starting from a given graded algebra. It is valid both for associative and Lie algebras. Thus, the term "an algebra" below means either an associative or a Lie algebra.

The initial data for the construction are: 1) a graded algebra $\Sigma_0 = (S, \Omega_0)$; 2) an Abelian group H and its epimorphism $\pi : H \to H_0$ onto the grading group H_0 of Σ_0. The tensor product $\mathbb{C}[H] \otimes S_0$ has a structure of the graded algebra: its grading subspaces are $\langle x \rangle \otimes V_y$, where $x \in H$, $\langle x \rangle$ is a one-dimensional linear subspace generated by x in the group algebra $\mathbb{C}[H]$, and V_y is a grading subspace of S_0 corresponding to a root $y \in \Omega_0$. Denote by Ω the following set of grading subspaces: $\Omega = \{\langle x \rangle \otimes V_{\pi(x)} \mid \pi(x) \in \Omega_0\}$; denote by S the direct sum of all linear subspaces from Ω. Evidently, S is the graded subalgebra of $\mathbb{C}[H] \otimes S$. We call it the induced graded algebra and denote by $\mathbb{C}[H] \otimes_\pi \Sigma_0$.

Lemma 3 *Let* $\Sigma = \mathbb{C}[H] \otimes_\pi \Sigma_0$ *be the induced graded algebra. Then*
(1) Σ *is finite-dimensional iff* Σ_0 *is finite-dimensional and* $\ker \pi$ *is finite;*
(1) Σ *is unrefinable, simple graded algebra iff* Σ_0 *is unrefinable, simple graded algebra.*

Proof of the lemma is the trivial exercise in algebra. □

There exists the equivalent definition of the induced graded algebra, which substantiates the use of the word "induced". Let $\Sigma_0 = (S_0, \Omega_0)$ be a graded algebra, A_0 be the group of all diagonal automorphisms of Σ_0 and $\iota : A_0 \to A$ be a monomorphism into a commutative group A. Denote by S the following algebra of functions: $S = \{F : A \to S_0 \mid F(xa_0) = a_0(F(x)) \text{ for } a_0 \in A_0\}$. Evidently, the group A acts as a diagonalizable subgroup of automorphisms of S. Therefore, S has a structure of a graded algebra, which we denote by Σ.

Lemma 4 *The graded algebra* Σ *is isomorphic to the induced algebra:* $\Sigma = \mathbb{C}[H] \otimes_\pi \Sigma_0$, *where* $H = A^\#$ *and* $\pi = \iota^\#$.

Proof is obtained as the reformulation of the definition of the induced algebra in the dual terms. □

3.5 Diagonalizable subgroups of $\mathrm{PGL}(N)$

All maximal diagonalizable subgroups of the group $G = \mathrm{PGL}(N)$ are well-known. In this subsection we describe them without proofs.

Let us fix a primitive root ε of unit of order m. Denote by X and Y $m \times m$ matrices

$$X = \begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \varepsilon^{m-1} \end{pmatrix}$$

Evidently, the matrices X, Y are nondegenerate. Denote by $\widehat{A}(m)$ the group, generated by X, Y. It is a subgroup of $\mathrm{GL}(m)$. Denote by $A(m)$ the image of $\widehat{A}$ under the projection $\mathrm{GL}(m) \to \mathrm{PGL}(m)$.

Lemma 5 $A(m) = \mathbb{Z}_m \times \mathbb{Z}_m$. *The subgroup $A(m)$ is a maximal diagonalizable subgroup of* $\mathrm{PGL}(m)$.

Subgroups $A(m)$ and tori are the bricks for constructing all diagonalizable subgroups of $\mathrm{PGL}(N)$. Let us factorize the number N as follows: $N = m_1 \dots m_s n$, where $m_i = p_i^{\alpha_i}$ is a power of a prime. Denote by K the subgroup of $\mathrm{PGL}(N)$, which is the image of the linear group $\mathrm{GL}(m_1) \times \dots \times \mathrm{GL}(m_s) \times \mathrm{GL}(n)$. Evidently, $K = \mathrm{PGL}(m_1) \times \dots \times \mathrm{PGL}(m_s) \times \mathrm{PGL}(n)$. The subgroup $A(m_i)$ of i-th factor $\mathrm{PGL}(m_i)$ we will consider also as a subgroup of $\mathrm{PGL}(N)$.

Proposition 4 .
(1) *An arbitrary diagonalizable subgroup of* $\mathrm{PGL}(N)$ *is isomorphic to a subgroup* $A = A(m_1) \times \dots \times A(m_s) \times A'$ *for an appropriate factorization* $N = m_1 \dots m_s n$ *and a subgroup* A' *of a maximal torus of* $\mathrm{PGL}(n)$*;*
(2) *A is maximal diagonalizable subgroup of* $\mathrm{PGL}(N)$ *iff A' coincides with a maximal torus of* $\mathrm{PGL}(n)$.

Let us accept the standard denotation: $A(m_1, \dots, m_s) = A(m_1) \times \dots \times A(m_s)$ is the subgroup of $\mathrm{PGL}(m_1) \times \dots \times \mathrm{PGL}(m_s)$ and also a subgroup of $\mathrm{PGL}(N)$ for $N = m_1 \dots m_s n$.

Note, that the subgroups $A(m_1, \dots, m_s)$ was know already to C.Jordan. They played the essential role in his classical work [6].

3.6 Proof of the theorem 1

Let $\Sigma = (S, \Omega)$ be a finite-dimensional, unrefinable, simple graded associative algebra.

Assume first, that S is the simple algebra. By Wedderburn's theorem, S is isomorphic to a matrix algebra M_N over the field of complex numbers. By the analog of the proposition 1 for associative algebras, it is sufficient to classify maximal diagonalizable subgroups of the group $\mathrm{Aut}\,\mathrm{M}_N = \mathrm{PGL}(N)$. It is done in the proposition 4. Therefore, all we need is to reformulate the properties of known diagonalizable subgroups in terms of gradings, defined by them. Thus, we get that in this case $\Sigma = \Sigma_{H,\xi} \otimes \mathrm{M}_n^c$. Certain details about unrefinable gradings of M_N are given in the further subsection.

Let us turn to the case of an unrefinable, simple graded algebra $\Sigma = (S, \Omega)$ with not simple underlying algebra S.

Lemma 6 *The underlying algebra S of an unrefinable, simple graded associative algebra $\Sigma = (S, \Omega)$ is the direct sum of algebras, each one isomorphic to the same simple algebra S_0.*

Proof is standard. □

Thus, by the lemma 6, $S = S_0 \oplus \dots \oplus S_0$, where S_0 is a simple algebra. Let A be the group of all diagonal automorphisms of Σ and $A_0 = \{a \in A \mid a(S_0) = S_0\}$. Then, clearly, A_0 is the diagonalizable subgroup of the group of all automorphisms of the algebra S_0. Therefore, A_0 defines the grading of the algebra S_0. Denote this graded algebra by $\Sigma_0 = (S_0, \Omega_0)$. Using the lemma 4, it it easy to show, that the graded algebra Σ is induced from the graded algebra Σ_0. By proposition 3, Σ_0 is unrefinable, simple graded algebra; its underlying algebra S_0 is simple associative algebra. Such algebras are already classified. Thus, $\Sigma_0 = \Sigma_{H_1,\xi} \otimes \mathrm{M}_n^c$ for an appropriate 2-cocycle ξ on an Abelian group H_1 and n.

Lemma 7 *Suppose,* $\Sigma_0 = \Sigma_{H_1,\xi} \otimes \mathrm{M}_n^c$ *and the underlying algebra of* Σ_0 *is* M_N. *Denote by* H_0 *the grading group of* Σ_0. *Assume,* Σ *is an induced graded algebra:* $\Sigma = \mathbb{C}\,[H] \otimes_\pi \Sigma_0$. *Then*
(1) $H_0 = H_1 \oplus \mathbb{Z}^{\,n-1}$;
(2) *the grading group* $H_{gr}(\Sigma)$ *coincides with* H;
(3) *there exists a decomposition* $H = \widehat{H}_1 \oplus H_2$ *such that* $H_2 = \mathbb{Z}^{\,n-1}$ *and the kernel of the homomorphism* π *lies in* $\widehat{H}_1$;
(4) $\Sigma = \Sigma_{\widehat{H}_1,\theta} \otimes \mathrm{M}_n^c$, *where the cocycle* θ *is defined by formula:* $\theta(x,y) = \xi(\pi(x),\pi(y))$.

We omit the proof, which is elementary. □

Thus, the proof of the theorem follows from this lemma. □

3.7 Unrefinable gradings of a matrix algebra

In this section we select those unrefinable, simple graded algebras, which has a simple (therefore, matrix) underlying algebra, and study them in more details.

The cohomology group $\mathrm{H}^2(H,\mathbb{C}^*)$ is well-known. We describe it shortly without proofs. For a cocycle $\xi \in \mathrm{Z}(H,\mathbb{C}^*)$ let us define a function $\langle x,y\rangle_\xi = \xi(x,y)\xi(y,x)^{-1}$ $(x,y \in H)$. The function $\langle x,y\rangle_\xi : H\times H \to \mathbb{C}^*$ is the analog of a skew-symmetric bilinear form on H. We call it 'the biexponential form' because the operation in $\mathbb{C}^*$ is the multiplication and operation in H is the addition.

Definition 4 *A map* $\langle x,y\rangle : H\times H \to \mathbb{C}^*$ *is called biexponential form iff* $\langle x+y,z\rangle = \langle x,z\rangle\cdot\langle y,z\rangle$ *and* $\langle x,y+z\rangle = \langle x,y\rangle\cdot\langle x,z\rangle$. *A biexponential form is called skew-symmetric iff* $\langle x,y\rangle = \langle y,x\rangle^{-1}$.

Lemma 8 *Let* $k = \mathbb{C}$. *Then (1) the map* $\xi \to \langle x,y\rangle_\xi$ *correctly defines the isomorphism of the group* $\mathrm{H}^2(H,\mathbb{C}^*)$ *to the group of all biexponential forms on* H *with respect to natural multiplication of forms* $\langle x,y\rangle'\cdot\langle x,y\rangle''$; (2) *cohomology classes* $[\xi']$ *and* $[\xi'']$ *belong to the same orbit of the group* $\mathrm{Aut}\,H$ *iff the corresponding forms* $\langle x,y\rangle_{\xi'}$ *and* $\langle x,y\rangle_{\xi''}$ *are equivalent as forms* [2].

Definition 5 *A 2-cocycle* ξ *is called non-degenerate iff the form* $\langle x,y\rangle_\xi$ *is non-degenerate.*

Proposition 5 *The underlying algebra* S *of a graded associative algebra* $\Sigma_{H,\xi} = (S,\mathcal{H})$ *is simple algebra (therefore matrix algebra) iff the cocycle* ξ *is nondegenerate.*

A biexponential skew-symmetric form can be represented in a canonical form which is an analog of the canonical form of a bilinear skew-symmetric form.

Proposition 6 *Let* $\langle x,y\rangle$ *be a nondegenerate biexponential form on a finite Abelian group* H. *For every* m *fix a primitive root* $\varepsilon_m \in \sqrt[m]{1}$. *Then there exists a minimal set of generators* $a_1,b_1,a_2,b_2,\ldots,a_s,b_s$ *of* H *which satisfies conditions:*
(1) *the elements* a_i *and* b_i *are of the same order* m_i *for* $i=1,\ldots,s$;
(2) $\langle a_i,b_i\rangle = \varepsilon_{m_i}$, $i=1,\ldots,s$; $\langle a_i,b_j\rangle = 0$, $i\neq j$; $\langle a_i,a_j\rangle = \langle b_i,b_j\rangle = 0$ *for any* i,j;
(3) *numbers* m_i *are powers of primes:* $m_i = p_i^{\alpha_i}$;
(4) *two biexponential forms are equivalent iff they have the same sets of numbers* $\{m_1,\ldots,m_s\}$.

Denote by $\mathrm{M}^g_{(m_1,\ldots,m_s)}$ the graded associative algebra, which corresponds to the biexponential form with given set of numbers $\{m_1,\ldots,m_s\}$. If $s=1$ then the underlying algebra of $\mathrm{M}^g_{(m)}$ is isomorphic to the matrix algebra M_m of order $m = p^\alpha$. Let us find grading subspaces in matrix term.

For any element X of the linear group $\widehat{A}(m) \subset \mathrm{GL}(m)$, which is defined in subsection 3.5, denote by $\langle X\rangle$ the one-dimensional linear subspace, generated by the matrix X. Evidently, we get m^2 different linear subspaces. Let us denote the set of all these linear subspaces by $\mathcal{H}_m$.

Lemma 9 *The set of subspaces* $\mathcal{H}_m$ *defines the structure of a graded algebra. The graded algebra* $(\mathrm{M}_m,\mathcal{H}_m)$ *is isomorphic to the graded algebra* M^g_m.

Corollary 1 (1) $\mathrm{M}^g_{(m_1,\ldots,m_s)} = \mathrm{M}^g_{m_1}\otimes\ldots\otimes\mathrm{M}^g_{m_s}$; (2) *an arbitrary finite-dimensional, unrefinable, simple graded associative algebra* $\Sigma = (S,\Omega)$, *such that* $S = \mathrm{M}_N$, *is isomorphic to a graded algebra* $\mathrm{M}^g_{m_1}\otimes\ldots\otimes\mathrm{M}^g_{m_s}\otimes\mathrm{M}^c_n$ *for an appropriate factorization* $N = m_1\ldots m_s n$ *and* m_i *be powers of primes:* $m_i = p_i^{\alpha_i}$.

[2] We use the terminology of the linear algebra in the case of finitely-generated Abelian groups; the meanings of terms are clear from the context.

4 Unrefinable, simple graded Lie algebras with not simple underlying Lie algebra

The construction of the induced graded Lie algebra allows to reduce the classification of unrefinable, simple graded Lie algebras to the case of graded algebras with simple underlying Lie algebra.

Theorem 2 *Let $\Gamma = (\mathfrak{g}, \Omega)$ be a finite-dimensional, unrefinable, simple graded Lie algebra. Assume that $\mathfrak{g}$ is not a simple Lie algebra. Then there exists an unrefinable, simple graded Lie algebra $\Gamma_0 = (\mathfrak{g}_0, \Omega_0)$ with simple underlying Lie algebra $\mathfrak{g}_0$, such that $\Gamma = \mathbb{C}[H] \otimes_\pi \Gamma_0$.*

Proof is exactly the same as the proof of the analogous fact for graded associative algebras, which is given in the subsection 3.6. □

5 The classical unrefinable, simple graded Lie algebras

¿From this section, we suppose that the underlying Lie algebra of a graded Lie algebra is a simple algebra.

5.1 1-st series: graded Lie algebras, associated to graded associative algebras

Let $\Sigma = (S, \Omega)$ be a graded associative algebra. Denote by $\mathrm{Lie}(S)$ the Lie algebra on the linear space S with the commutator $[X, Y] = XY - YX$. Evidently, the set Ω of linear subspaces defines the structure of graded Lie algebra [3] on $\mathrm{Lie}(S)$. We denote this graded Lie algebra by $\mathrm{Lie}(\Sigma)$ and call it the graded Lie algebra associated with Σ.

The center C of an underlying Lie algebra $\mathfrak{g}$ of a graded Lie algebra $\Gamma = (\mathfrak{g}, \Omega)$ is a graded ideal of Γ. Therefore, the factor algebra Γ/C inherits the structure of a graded Lie algebra.

Proposition 7 *Let $\Sigma = (S, \Omega)$ be an unrefinable graded associative algebra with simple underlying algebra, i.e. S is isomorphic to a matrix algebra M_N. Then the algebra $\Gamma = \mathrm{Lie}(\Sigma)/C$ (C is the center of $\mathrm{Lie}(\Sigma)$) is an unrefinable, simple graded Lie algebra.*

We omit the elementary proof of this proposition. □

By the theorm 1 and the corollary 1, an arbitrary finite-dimensional, unrefinable graded algebra, which underlying algebra is a matrix algebra, is isomorphic to $\Sigma = \mathrm{M}^g_{(m_1,\dots,m_s)} \otimes \mathrm{M}^c_n$. Denote the graded Lie algebra $\mathrm{Lie}(\Sigma)/C$ by $A^1_{(m_1,\dots,m_s);n-1}$. We accept certain restrictions on the parameters of this series: $s = 0$ and $n \geq 1$; or $s = 1$ and ($n = 0$ or $n \geq 2$); or $s > 1$ and $n \geq 2$.

By proposition 7, Σ is finite-dimensional, unrefinable, simple graded Lie algebra. In this subsection we compute main invariants of graded Lie algebras from this first series.

Assume first, that $\Sigma = \mathrm{M}^g_{(m_1,\dots,m_s)}$. By the results of the subsection 3.7, Σ is a generalized Clifford algebra with nondegenerate 2-cocycle ξ: $\Sigma = \Sigma_{H,\xi}$.

Denote by $\Gamma^g_{(m_1,\dots,m_s)}$ the graded Lie algebra $\mathrm{Lie}(\mathrm{M}^g_{(m_1,\dots,m_s)})$. By definition, the set of elements $\{x \mid x \in H\}$ form a basis of the generalized Clifford algebra. In order to distinguish elements of basis of Lie algebra and elements of the Abelian group H we denote the elements of this basis by $E_x, x \in H$. Evidently, the set Ω of grading subspaces of the factor-algebra $\Gamma = \Gamma^g_{(m_1,\dots,m_s)}/C$ can be identified with $H \setminus \{0\}$ (zero is deleted because of factorization by center).

Lemma 10 *The structure constants of the graded Lie algebra $\Gamma = \Gamma^g_{(m_1,\dots,m_s)}/C$ in the basis $\{E_x \mid x \in H\setminus\{0\}\}$ are given by the formula $[E_x, E_y] = f(x,y)E_{x+y}$ $\forall x, y \in H\setminus\{0\}$, where $f(x,y) = \xi(x,y) - \xi(y,x)$. Particularly, $[E_x, E_y] = 0$ iff $\langle x, y\rangle_\xi = 1$.*

Proof is evident. □

Denote by ρ the tautological embedding of $\Omega = H \setminus \{0\}$ into the grading group H_{gr} of Γ and put $\rho(0) = 0$, by definition. If $[E_x, E_y] \neq 0$ then $\rho(x + y) = \rho(x) + \rho(y)$. Therefore, ρ is the 'partial homomorphism' of the group H to H_{gr}. Although ρ is a bijection of sets and the partial homomorphism, nevertheless, in certain cases it is not a group isomorphism.

We distinguish two cases: 1) at least one number among m_i is greater than 2, or s=1 and $m_1 = 2$; 2) $s > 1$ and $m_1 = \ldots = m_s = 2$. In the last case, we denote s-component vector $(2, \ldots, 2)$ by $\mathbf{2}(s)$ for shortness.

[3] The commutativity of a grading group H_{gr} of a graded associative algebra is essential just at this point

Proposition 8 *Assume, that there is at least one* $m_i > 2$ *or* $s = 1, m_1 = 2$. *Let* $N = m_1 \dots m_s$. *Then the invariants of* Γ *are as follows:*
(1) $\rho : H \rightarrow H_{gr}$ *is an isomorphism of the groups;*
(2) $\rho(\Omega) = H_{gr} \setminus \{0\}$;
(3) *the group* $\widehat{W}$ *consists of all automorphisms of* H, *which preserve biexponential form* $\langle x, y\rangle_\xi$ *or transfer it to the inverse form:* $\widehat{W} = \{g \in \mathrm{Aut}\, H \mid \langle g(x), g(y)\rangle_\xi = \langle x, y\rangle_\xi^{\pm 1}\}$;
(4) *the underlying Lie algebra is* $\mathfrak{g} = \mathfrak{sl}\,(N)$, *its automorphism group* $G = \mathbb{Z}_2 \cdot \mathrm{PGL}(N)$;
(5) *the subgroup* $A(\Gamma)$ *coincides with the group* $A(m_1, \dots, m_s) \subset \mathrm{PGL}(N)$, *defined in the subsection 3.5.*

We omit the simple proof of the proposition. □

In the case 2) we identify the Abelian group H with the additive group of a vector space $\mathbb{F}_2^{2s}$ over the two-element field $\mathbb{F}_2$. It can be easily checked, that $\langle x, y\rangle_\xi = (-1)^{(x,y)}$, where (x, y) is a nondegenerate bilinear form over the field $\mathbb{F}_2$. It is known, that $(x, y) = q(x + y) + q(x) + q(y)$ for a certain function $q(x) \in \mathbb{F}_2$, which is called quadratic form. Thus, $\langle x, y\rangle_\xi = (-1)^{q(x+y)+q(x)+q(y)}$.

The invariants of Γ in this case are described in the proposition below.

Proposition 9 *Let* $s > 1$ *and* $m_1 = \dots = m_s = 2$. *Then*
(1) $H_{gr} = \mathbb{F}_2 \oplus H$, $H = \mathbb{F}_2 \oplus \mathbb{F}_2^{2s}$ *and* ρ *is given by formula:* $\rho(x) = (q(x) + 1)\omega + x$, *where* ω *is the generator of the direct summand* $\mathbb{F}_2$;
(2) $\rho(\Omega) = \{(q(x) + 1)\omega + x \mid x \in H, x \neq 0\}$;
(3) *the group* $\widehat{W}$ *is isomorphic (as an abstract group) to the symplectic group* $\mathrm{Sp}(H)$ *of the form* $\langle x, y\rangle_\xi$;
(4) *the action* π *of* $\widehat{W} = \mathrm{Sp}(H)$ *by automorphisms of* H_{gr} *is given by the formula:* $\pi(g)(\omega) = \omega$, $\pi(g)(x) = (q(g(x)) + q(x))\omega + g(x)$ $(g \in \widehat{W})$;
(5) *the underlying Lie algebra is* $\mathfrak{g} = \mathfrak{sl}\,(N)$, $N = 2^{2s}$;
(6) *the subgroup* $A(\Gamma)$ *is the direct product of the subgroup* $A(2, \dots, 2)$ *of* $\mathrm{PGL}(N)$ *and a two-element subgroup, generated by an external automorphism of the Lie algebra* $\mathfrak{g} = \mathfrak{sl}\,(N)$;

Proof is reduced to the direct algebraic verification; we miss it here. □

Thus, in the case of the generalized Clifford algebra the invariants are described.

Assume now, that Σ is the Cartan grading of a matrix algebra: $\Sigma = \mathrm{M}_n^c$. It is clear, that in this case the graded Lie algebra $\Gamma = \mathrm{Lie}(\mathrm{M}_n^c)/C$ is just Cartan root decomposition of $\mathfrak{sl}_n$. Thus, we obtain the proposition:

Proposition 10 *Let* $\Gamma = \mathrm{Lie}(\mathrm{M}_n^c)/C$. *Then*
(1) $H_{gr} = \mathbb{Z}^{n-1}$;
(2) $\rho(\Omega) = \{e_i - e_j \mid i \neq j, i, j = 1, \dots, n\} \sqcup \{0\}$, *where elements* e_i *generates* H_{gr} *and* $e_1 + \dots + e_m = 0$ *(in other words,* $\rho(\Omega)$ *is the classical root system* A_{n-1} *enlarged by zero);*
(3) *the group* $\widehat{W}$ *is generated by the Weyl group* $W = S_n$ *of type* A_{n-1} *and the automorphism* τ, *which acts on* H_{gr} *as* -1;
(4) $A(\Gamma)$ *is maximal torus* T^{n-1} *of the group* $\mathrm{PGL}(n)$.

Let us turn to the general case $\Gamma = A^1_{(m_1,\dots,m_s);n-1}$. The invariants for the cases $n = 0$ (a generalized Clifford algebra) and $s = 0$ (a Cartan root decomposition $A^1_{\emptyset;n} = A_n$) are described above.

Proposition 11 *Let* $s > 0$, $n > 0$ *and* Γ *be of the type* $A^1_{(m_1,\dots,m_s);n}$. *Then*
(1) $H_{gr} = H' \oplus H''$, *where* $H' = \bigoplus_{i=1}^{s}(\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i})$, $H'' = H_{gr}(A_n) = \mathbb{Z}^n$;
(2) $\rho(\Omega) = \{x + \alpha \mid x \in H', \alpha \in \Omega'' \subset H''\}$ *where* $\Omega'' = \Omega(A_n)$ *is the root system of* $A^1_{\emptyset;n}$;
(3) *the group* $\widehat{W}$ *is generated by the subgroup* W *of index* 2 *and an element* τ; $W = W' \oplus W''$ *where* W' *is the group of all authomorphisms of* H' *preserving the biexponential form* $\langle x, y\rangle_\xi$, $W'' = W(A_n) = S_{n+1}$ *is the Weyl group of type* A_n; τ *preserves* H' *and* H'', $\tau(x) = -x$ *if* $x \in H''$, $\tau|_{H'}$ *transfers the form* $\langle x, y\rangle$ *into inverse one:* $\langle \tau(x), \tau(y)\rangle_\xi = \langle x, y\rangle_\xi^{-1}$;
(4) *the underlying Lie algebra* $\mathfrak{g} = \mathfrak{sl}\,(N)$, $N = m_1 \dots m_s(n + 1)$;
(5) *the subgroup* $A(\Gamma)$ *coinsides with* $A(m_1, \dots, m_s) \times A''$, *where* A'' *is maximal torus* T^n *of the factor* $\mathrm{PGL}(n + 1)$ *of the subgroup* $K = \mathrm{PGL}(m_1) \times \dots \times \mathrm{PGL}(m_s) \times \mathrm{PGL}(n + 1) \subset G$.

The proof can be derived from the definition of the subgroup $A(m_1, \dots, m_s)$ and propositions 8,9,10. □

5.2 2-nd series: gradings of Lie algebra $\mathfrak{sl}_n$ which are not associated to graded associative algebras

Algebras from this series are tensor products (in special meaning) of certain graded Lie algebras and certain noncommutative graded associative algebras. First, we define the factor which is a graded Lie algebra.

Initial data for the construction of the graded Lie algebra are an n-element set F and an involution $\tau : F \to F$. Denote by I the set of fixed points of τ. Let us devide the complement $F \setminus I$ into two nonintersecting sets J and J', such that $\tau(J) = J', \tau(J') = J$. Let $\#I = k$ and $\#J = \#J' = m$. Thus, $n = k + 2m$. We use letters s, t to denote an arbitrary element of F, i for elements of I , j for elements of J, j' for elements of J'.

Identify F with $\{1, 2, \ldots, n\}$ and denote by $E_{s,t}$ $(s, t \in F)$ an elementary matrix of order $n \times n$. We will write also $E_{i,j}$ for $i \in I, j \in J$ and so on, because I, J, J' are subsets of F.

Define matrix $X^\sigma_{s,t}$, where $s, t \in F, \sigma \in \mathbb{F}_2$, by the formula: $X^\sigma_{s,t} = E_{s,t} - (-1)^\sigma E_{\tau(t),\tau(s)}$.

Lemma 11 *Matrices $X^\sigma_{s,t}$ satisfy the conditions:*
(1) $X^\sigma_{s,t} = 0$ *iff* $t = \tau(s)$ *and* $\sigma = 0$;
(2) $X^\sigma_{\tau(t),\tau(s)} = (-1)^{\sigma+1} X^\sigma_{s,t}$ *for all* $s, t \in F$, $\sigma \in \mathbb{F}_2$;
(3) *if* $s \neq k$, $X^{\sigma_1}_{s,t} \neq 0, X^{\sigma_2}_{t,k} \neq 0$ *then* $[X^{\sigma_1}_{s,t}, X^{\sigma_2}_{t,k}] = \lambda X^{\sigma_1+\sigma_2}_{s,k}$ *where*

$$\lambda = \begin{cases} 1 & \text{if } t \neq \tau(s), t \neq \tau(k) \\ 2 & \text{if } t = \tau(s) \text{ or } t = \tau(k) \end{cases}$$

(4) *if* $X^{\sigma_1}_{s,t} \neq 0, X^{\sigma_2}_{t,s} \neq 0$, *then* $[X^{\sigma_1}_{s,t}, X^{\sigma_2}_{t,s}] = \lambda(X^{\sigma_1+\sigma_2}_{s,s} - X^{\sigma_1+\sigma_2}_{t,t})$ *where*

$$\lambda = \begin{cases} 1 & \text{if } t \neq \tau(s) \\ 2 & \text{if } t = \tau(s) \end{cases}$$

(5) *if* $t \neq l, t \neq \tau(k), s \neq \tau(l), s \neq k$, *then* $[X^{\sigma_1}_{s,t}, X^{\sigma_2}_{l,k}] = 0$

Proof is trivial. □

Let us define certain linear subspaces of $\mathfrak{gl}_n$. For $s, t \in F, s \neq t$ and $\sigma \in \mathbb{F}_2$ define $V^\sigma_{s,t} = \langle X^\sigma_{s,t} \rangle$; $V^0_0 = \langle X^0_{j,j} \mid j \in J \rangle$ and $V^1_0 = \langle X^1_{i,i},\ X^1_{j,j} \mid i \in I, j \in J \rangle$.

It follows from the lemma 11, that some of these linear subspaces coincide. Namely, $V^\sigma_{s,t} = V^\sigma_{\tau(t),\tau(s)}$. Some of them are zero, namely, $V^0_{j,\tau(j)} = V^0_{\tau(j),j} = \{0\}$ for $j \in J$. Denote by Ω the set $\{V^\sigma_c \mid \sigma \in \mathbb{F}_2, c = 0$ or $c = (s,t) = (\tau(t), \tau(s), s \neq t\}$ of all this linear subspaces, except zero one (coinciding subspaces define one element of Ω).

Proposition 12 *Ω defines the structure of a graded Lie algebra on $\mathfrak{gl}_n$. Denote this graded Lie algebra by $\Gamma^e_{(k,m)}$ and assume in addition that $n \geq 2$. Then factor-algebra $\Gamma = \Gamma^e_{(k,m)}/C$ by the center is an unrefinable graded Lie algebra.*

Proof. Denote by H an Abelian group with generators ω, ε_i $(i \in I)$, e_j $(j \in J)$ and defining relations as follows: $H = \langle \omega, \varepsilon_i, e_j \mid 2\omega = 2\varepsilon_i = 0 \rangle$. Thus, $H = \mathbb{F}_2 \oplus \mathbb{F}_2^k \oplus \mathbb{Z}^m$.

For any $s \in F$ define an element $\epsilon_s \in H$ by formula:

$$\epsilon_s = \begin{cases} \varepsilon_i & \text{if } s = i \in I \\ e_j & \text{if } s = j \in J \\ -e_{\tau(j')} & \text{if } s = j' \in J' \end{cases}$$

Define a map ρ of Ω into H by formulas: $\rho(V^0_0) = 0$, $\rho(V^1_0) = \omega$, $\rho(V^\sigma_{s,t}) = \epsilon_s - \epsilon_t + \sigma\omega$. The definition of ρ is correct because the identity $\epsilon_{\tau(t)} - \epsilon_{\tau(s)} = \epsilon_s - \epsilon_t$ holds in H. Evidently, ρ is an embedding of Ω into H. To compete the proof we need lemma:

Lemma 12 *The following identities holds for the commutants of linear subspaces $V \in \Omega$:*
(1) $[V^0_0, V^0_0] = [V^0_0, V^1_0] = [V^1_0, V^1_0] = \{0\}$;
(2) $[V^0_0, V^\sigma_{s,t}] \subset V^\sigma_{s,t}$, $[V^1_0, V^\sigma_{s,t}] \subset V^{\sigma+1}_{s,t}$;
(3) *if* $V_1, V_2 \in \Omega$ *and* $V_1 = V^{\sigma_1}_{s,t}$, $V_2 = V^{\sigma_2}_{l,k}$, *then* $[V_1, V_2] = \{0\} \Leftrightarrow \rho(V_1) + \rho(V_2) \notin \Omega$; $[V_1, V_2] = V_3 \Leftrightarrow \rho(V_1) + \rho(V_2) = \rho(V_3)$.

We skip the trivial proof of this lemma. □

It follows from the lemma, that for any $V_1, V_2 \in \Omega$ either $[V_1, V_2] = \{0\}$ or $[V_1, V_2] \subset V_3$ and $\rho(V_1) + \rho(V_2) = \rho(V_3)$. Therefore, $\Gamma^e_{(k,m)}$ is a graded Lie algebra. Graded Lie algebra $\Gamma = \Gamma^e_{(k,m)}/C$ is a simple one because the underlying Lie algebra is simple algebra. It is an easy exercise to check, that Γ is unrefinable graded algebra.□

Denote by S_k the symmetric group of rang k and by $W(BC_m)$ the Weyl group of the classical root system BC_m (it coincides with Weyl groups of both B_m and C_m). The group $W(BC_m)$ coincides with the group of all linear transformations, which permute the set $\{\pm e_i\}$ where e_i form a basis of linear space. If $m = 1$ then $W(BC_m) = \mathbb{Z}_2$. Let us accept the agreement that $W(BC_0) = \{e\}$ and also $S_0 = S_1 = \{e\}$, $\mathbb{Z}_2^0 = \mathbb{Z}_2^{-1} = \{e\}$.

Proposition 13 *Assume that* $n = k+2m > 2$. *Invariants of the graded Lie algebra* $\Gamma^e_{(k,m)}$ *are as follows:*
(1) $H_{gr}(\Gamma^e_{(k,m)}) = H' \oplus H_0$ *where* $H' = \{x \in H \mid x = \sum \lambda_i \varepsilon_i + \sum \mu_j e_j,\ \sum \lambda_i + \sum \mu_j (\mathrm{mod}\, 2) = 0$ *(in* $\mathbb{F}_2$*)*$\}$, $H_0 = \langle \omega \rangle \subset H$;
(2) $\Omega(\Gamma^e_{(k,m)}) = \{\sigma\omega, \varepsilon_{i_1} + \varepsilon_{i_2} + \sigma\omega, \varepsilon_i \pm e_j + \sigma\omega, \pm e_{j_1} \pm e_{j_2} + \sigma\omega, \pm 2e_j + \omega \mid \sigma \in \mathbb{F}_2; i, i_1, i_2 \in I, i_1 \neq i_2; j, j_1, j_2 \in J, j_1 \neq j_2\}$;
(3) *the group* $\widehat{W}(\Gamma^e_{(k,m)}) \subset \mathrm{Aut}\, H_{gr}$ *is isomorphic to the semidirect product of the groups* $W_1 = S_k \times W(BC_m)$ *and* $W_0 = \mathbb{Z}_2^{k-1} \times \mathbb{Z}_2^m$ *(which is normal subgroup); the group* S_k *acts by permutations of elements* ε_i*; the group* $W(BC_m)$ *acts naturally on the subgroup* $\mathbb{Z}^m = \langle e_j \rangle$ *and* $W_0 = \{g \in \mathrm{Aut}\, H_{gr} \mid g(x) - x \in \langle \omega \rangle\ \forall x \in H_{gr}\}$;
(4) *subgroup* $A(\Gamma)$ *is generated by its subgroup* A_0 *of index* 2 *and by automorphism* t *of order* 2*;* A_0 *is contained in the subgroup* $P(\mathrm{GL}(k) \oplus \mathrm{GL}(2m))$ *of the connected component of identity* $G^\circ = \mathrm{PGL}(n)$*;* $A_0 = A_0' \cdot A_0''$ *and* $A_0' \subset \mathrm{GL}(k)$ *consists of all diagonal matrices with* ± 1 *on the diagonal;* $A_0'' \subset \mathrm{GL}(2m)$ *is* m*-dimensional torus consisting of all diagonal matrices, for which the products of* j*-th and* $m+j$*-th diagonal elements are equal to* 1 *(*$j = 1, \ldots, m$*);* t *is an external automorphism of Lie algebra* $\mathfrak{gl}_n$ *of order* 2*, which commute with* A_0.

Proof follows from the direct calculations.□

The invariants of the factor-algebra $\Gamma^e_{(k,m)}/C$ is equal to invariants of $\Gamma^e_{(k,m)}$, except the case $m = 0$; in this case the only difference is that $0 \notin \Omega(\Gamma^e_{(k,m)})$.

Lie algebras $\Gamma^e_{(k,m)}$ have a certain property, which allows to define its tensor product with noncommutative graded associative algebra $\mathrm{M}^g_{(\mathbf{2(s)})}$. Denote by $\{X, Y\}$ the operation $XY + YX$ in a matrix algebra.

Proposition 14 *Let* $\Gamma^e_{(k,m)} = (\mathfrak{gl}_n, \Omega)$, $\alpha, \beta \in \rho(\Omega)$ *be roots and* $V_\alpha, V_\beta \in \Omega$ *be root subspaces. Then* $\{V_\alpha, V_\beta\} \subset V_{\alpha+\beta+\omega}$

Proof is evident. □

Define the tensor product of a graded associative algebra $\Sigma = \mathrm{M}^g_{(2,\ldots,2)} = (\mathbb{C}[H]_\xi, \mathcal{H})$ ($H = \mathbb{F}_2^{2s}$ and the cocycle ξ is nondegenerate) and a graded Lie algebra $\Gamma = \Gamma^e_{(k,m)} = (\mathfrak{gl}_n, \Omega)$, $n = k + 2m$. Both factors of the tensor product $\mathbb{C}[H]_\xi \otimes \mathfrak{gl}_n$ are equipped with the structure of an associative algebra. Therefore, the tensor product has the structure of Lie algebra, defined by commutator $[X, Y] = XY - YX$. Evidently, this Lie algebra is isomorphic to $\mathfrak{gl}_N$, $N = 2^s n$. Denote by $\mathcal{H} \otimes \Omega$ the set of linear subspaces $\{\langle x \rangle \otimes V \mid \langle x \rangle \in \mathcal{H}, V \in \Omega\}$.

Proposition 15 *The set of linear subspaces* $\mathcal{H} \otimes \Omega$ *defines a structure of graded Lie algebra on* $\mathfrak{gl}_N = \mathbb{C}[H]_\xi \otimes \mathfrak{gl}_n$. *Denote this graded Lie algebra by* $\mathrm{Lie}(\Sigma \otimes \Gamma)$. *The quotient algebra* $\mathrm{Lie}(\Sigma \otimes \Gamma)/C$ *by it center* C *is an unrefinable, simple graded Lie algebra.*

The proof of the proposition follows immediately from the lemma below.

Lemma 13 *Let* $x, y \in H$, E_x, E_y *be relevant basis vectors of* $\mathbb{C}[H]_\xi$, $X_\alpha \in V_\alpha, X_\beta \in V_\beta$, *where* $V_\alpha, V_\beta \in \Omega$. *Then* $[E_x \otimes X_\alpha, E_y \otimes X_\beta] =$
$= E_{x+y} \otimes \left(\frac{1}{2}(\xi(x,y) + \xi(y,x))[X_\alpha, X_\beta] + \frac{1}{2}(\xi(x,y) - \xi(y,x))\{X_\alpha, X_\beta\}\right)$.

Proof follows from the direct calculations. It is important, that in this case $\xi(x,y)\xi(y,x)^{-1} = \langle x, y \rangle \in \{\pm 1\}$).□

It follows from the lemma, that the commutant of linear subspaces $V_1, V_2 \in \mathcal{H} \otimes \Omega$ is contained in a linear subspace from $\mathcal{H} \otimes \Omega$. There is the natural embedding $\rho : \mathcal{H} \otimes \Omega \to H \oplus H_{gr}(\Gamma)$. Note that the commutator of linear subspaces from $\mathcal{H} \otimes \Omega$ do not corresponds to a sum of elements of the group.

Nevertheless, it can be easily shown, that partial operation on $H \oplus H_{gr}(\Gamma)$, defined by the mean of the commutator of linear subspaces, can be extended to a new group operation. Therefore, $\mathrm{Lie}(\Sigma \otimes \Gamma)$ is a graded Lie algebra. We omit the proof of unrefinability of its quotient algebra by center. □

We call the graded Lie algebra $\mathrm{Lie}(\mathrm{M}^g_{(\mathbf{2}(s))} \otimes \Gamma^e_{(k,m)})/C$, an algebra of the type $A^2_{\mathbf{2}(s);k;m}$. We suppose, that $s = 0$ and $k + 2m > 2$; or $s = 1$ and $k + 2m \geq 2$; or $s > 1$ and $k + 2m \geq 2$ or $k = 1, m = 0$.

Proposition 16 *Let $\Gamma = (\mathfrak{sl}_N, \Omega)$ be a graded Lie algebra of the type $A^2_{\mathbf{2}(s);k;m}$ and $s > 0$, $n = k+2m \geq 2$. Then*
(1) $H_{gr}(\Gamma) = H' \oplus H_0 \oplus H''$, where $H' \oplus H_0 = H_{gr}(\Gamma^e_{(k,m)})$, $H'' = \mathbb{F}_2^{2s}$ is an Abelian group, which is used in the definition of algebra $\Gamma^g_{(\mathbf{2}(s))}$;
(2) the map $\rho : \Omega \to H_{gr}(\Gamma)$ is given by the formula: $\rho(\langle x\rangle \otimes V_\alpha) = \alpha + q(x)\omega + x$ where $x \in H''$, $q(x)$ is fixed quadratic form on H'' (see proposition 9);
(3) $\rho(\Omega) = \{\alpha + x \mid x \in H'', \alpha \in \Omega_x \subset H' \oplus H_0\}$, where $\Omega_x = \{\sigma\omega, \varepsilon_{i_1} + \varepsilon_{i_2} + \sigma\omega, \varepsilon_i \pm e_j + \sigma\omega, \pm e_{j_1} \pm e_{j_2} + \sigma\omega, \pm 2e_j + q(x)\omega \mid \sigma \in \mathbb{F}_2; i, i_1, i_2 \in I, i_1 \neq i_2; j, j_1, j_2 \in J, j_1 \neq j_2\}$, if $m = 0$ the element $(0,0)$ is excluded from Ω_x;
(4) $\widehat{W}(\Gamma) = W' \times W''$, where $W' = \widehat{W}(\Gamma^e_{(k,m)})$, $W'' \approx \mathrm{Sp}(\mathbb{F}_2^{2s})$ and the action π of the group $\widehat{W}$ is as follows: for $g \in W''$ we have $\pi(g)(\alpha + x) = \alpha + (q(g(x) + q(x))\omega + g(x)$; for $g \in W'$ we have $\pi(g)(\alpha + x) = g(\alpha) + x$ ($\alpha \in H' \oplus H_0, x \in H''$);
(5) the subgroup $A(\Gamma)$ is generated by the index two subgroup A_0 and an element t of order two; $A_0 = A(2,\ldots,2) \times A'_0$, where $A(2,\ldots,2) \subset \mathrm{PGL}(2) \times \ldots \times \mathrm{PGL}(2)$ is a subgroup from the subsection 3.5 for $m_1 = \ldots = m_s = 2$; $A'_0 \subset \mathrm{PGL}(n)$ coinsides with the index two subgroup of the group of diagonal automorphisms of the graded Lie algebra $\Gamma^e_{(k,m)}$, A consists of inner automorphisms; t is an order two outer automorphism of the Lie algebra $\mathfrak{gl}_N$, which commut with any element of A_0.

We omit the proof, which is based on a standard algebraic technic. □

5.3 3-d and 4-th series: twisted analogs of the 2-nd series

The last two series are twisted analogs of $A^2_{\mathbf{2}(s);k;m}$.

In the proposition 9 the quadratic form $q(x)$ on the linear space $\mathbb{F}_2^{2s}$ over the field $\mathbb{F}_2$ was introduced. It is known, that there exists exactly two classes of nonequivalent nondegenerate quadratic forms over $\mathbb{F}_2$. The difference of two quadratic forms is a linear form. In the proposition 16 different quadratic forms leads to isomorphic root systems. It is not valid for twisted analogs. Let us fix the denotation $q(x)$ for the form $q(x) = \sum_{i=1}^s \lambda_{2i-1}\lambda_{2i}$.

For an arbitrary graded Lie algebra Γ and a subgroup H of its grading group $H_{gr}(\Gamma)$ one can define the graded subalgebra $\Gamma(H)$, which is the direct summ of all grading subspaces corresponding to roots lying in H.

We are interesting here in the case: $\Gamma = A^2_{\mathbf{2}(s);k;m}$ and H is a subgroup of $H_{gr}(\Gamma)$ of index 2. Therefore, H coincides with a kernel of some nontrivial homomorphism $t : H_{gr}(\Gamma) \to \mathbb{F}_2$. Homomorphism t we consider as a linear form on $H_{gr}(\Gamma)$ with values in $\mathbb{F}_2$.

Definition of two linear forms t_1 and t_2. We use denotations from proposition 16. We have $H_{gr}(\Gamma) = H' \oplus H_0 \oplus H''$. Define $t_1(H') = t_1(H'') = 0, t_1(\omega) = 1$.

Linear form t_2 is defined only for a special set of parameters s, k, m. For $s > 0$ define $t_2(H') = 0, t_2(\omega) = 0$ and $t_2|_{H''}$ be a fixed nonzero linear form such, that quadratic form $p(x) = q(x) + t_2(x)$, ($x \in H''$) is not equivalent to $q(x)$. In the case $s = 0$ and $k = 0$ the grading group $H_{gr}(\Gamma)$ is decomposed into the direct sum $H' \oplus H_0$, where $H' = \{\sum \mu_j e_j \mid \sum \mu_j(2) = 0\}$. Define $t_2(\omega) = 1$, $t_2(e_{j_1} - e_{j_2}) = 0$, $t_2(e_{j_1} + e_{j_2}) = 1$.

Lemma 14 *Let $t = t_1$ or $t = t_2$. Then*
(1) the kernel of t is generated by the roots, lying in it, and do not contains the root ω;
(2) an arbitrary linear form $t' : H_{gr}(\Gamma) \to \mathbb{F}_2$, which satisfies the condition (1), can be transformed either to t_1 or to t_2 by an automorphism from the group $\widehat{W}(\Gamma)$.

Proof follows from the description of the groups $H_{gr}(\Gamma)$ and $\widehat{W}(\Gamma)$ (see proposition 16). □

Denote by $BD_{\mathbf{2}(s);k;m}$ the subalgebra of $A^2_{\mathbf{2}(s);k;m}$, which is constructed with the use of t_1, and by $C_{\mathbf{2}(s);k;m}$ those one, which is constructed with the use of t_2.

We describe the invariants of $BD_{\mathbf{2}(s);k;m}$ and $C_{\mathbf{2}(s);k;m}$ in propositions below. We use the denotations of the proposition 16, particularly, the denotations of subgroups H', H_0, H'' in the decomposition $H_{gr}(A^2_{\mathbf{2}(s);k;m}) = H' \oplus H_0 \oplus H''$.

Proposition 17 *Let* $\Gamma = BD_{\mathbf{2}(s);k;m} = (\mathfrak{g}, \Omega)$. *Then*
(1) Γ *is unrefinable simple graded Lie algebra;*
(2) $\mathfrak{g} = \mathfrak{so}_N$ *where* $N = 2^s(k+2m)$;
(3) $H_{gr}(\Gamma) = H' \oplus H''$, *where* H', H'' *are subgroups of* $H_{gr}(A^2_{\mathbf{2}(s);k;m})$, *defined in proposition 16;*
(4) $\rho(\Omega) = \{\alpha + x \mid x \in H'', \alpha \in \Omega_x \subset H'\}$, *where* Ω_x *is defined as follows: define* $\Omega' = \{0, \varepsilon_{i_1} + \varepsilon_{i_2}, \varepsilon_i \pm e_j, \pm e_{j_1} \pm e_{j_2} \mid i, i_1, i_2 \in I, i_1 \neq i_2; j, j_1, j_2 \in J, j_1 \neq j_2\}$, *then* $\Omega_x = \Omega'$ *if* $q(x) \neq 0$; $\Omega_x = \Omega' \cup \{\pm 2e_j \mid j \in J\}$ *if* $q(x) = 0$ *and* $m \neq 0$ *or* $x \neq 0$; $\Omega_x = \Omega' \cup \{\pm 2e_j \mid j \in J\} \setminus \{0\}$ *if* $q(x) = 0$, $m = 0$ *and* $x = 0$;
(4) $\widehat{W}(\Gamma) = W_1 \times W_2$, *where* $W_1 = \mathrm{O}_q(\mathbb{F}_2^{2s})$ *is the orthogonal group of the quadratic form* $q(x)$, $W_2 = S_k \times W(BC_m)$ *and the action of* $W_1 \times W_2$ *is component-wise.*

For the linear form t_2 we have

Proposition 18 *Let* $\Gamma = C_{\mathbf{2}(s);k;m} = (\mathfrak{g}, \Omega)$. *Then*
(1) Γ *is unrefinable simple graded Lie algebra;*
(2) $\mathfrak{g} = \mathfrak{sp}_N$ *where* $N = 2^s(k+2m)$;
(3) $H_{gr}(\Gamma) = H' \oplus H'''$, *where the subgroup* H' *was defined in proposition 16,* $H''' = \{t_2(x)\omega + x \mid x \in H''\}$
(4) $\rho(\Omega) = \{\alpha + y \mid y \in H''', \alpha \in \Omega_y \subset H'\}$, *where* Ω_y *is defined as follows: define* $\Omega' = \{0, \varepsilon_{i_1} + \varepsilon_{i_2}, \varepsilon_i \pm e_j, \pm e_{j_1} \pm e_{j_2} \mid i, i_1, i_2 \in I, i_1 \neq i_2; j, j_1, j_2 \in J, j_1 \neq j_2\}$ *and* $p(y) = q(y) + t_2(y)$, *then* $\Omega_y = \Omega'$ *if* $p(y) \neq 0$; $\Omega_y = \Omega' \cup \{\pm 2e_j \mid j \in J\}$ *if* $p(y) = 0$ *and* $m \neq 0$ *or* $y \neq 0$; $\Omega_y = \Omega' \cup \{\pm 2e_j \mid j \in J\} \setminus \{0\}$ *if* $p(y) = 0$ *and* $m = 0, y = 0$;
(4) $\widehat{W}(\Gamma) = W_1 \times W_2$, *where* $W_1 = \mathrm{O}_p(\mathbb{F}_2^{2s})$ *is orthogonal group of the quadratic form* $p(x)$, $W_2 = S_k \times W(BC_m)$ *and the action of* $W_1 \times W_2$ *is component-wise.*

We omit the description of subgroups of diagonal automorphisms $A(\Gamma) \subset G$ for these twisted series. It can be easily derived from the proposition 16.

5.4 Classification of unrefinable gradings of simple Lie algebras of the classical type.

In this section we show that all unrefinable gradings of finite-dimensional simple Lie algebras of classical type are grading from the series defined above.

Theorem 3 *Assume that* $\Gamma = (\mathfrak{g}, \Omega)$ *is a finite-dimensional, unrefinable graded Lie algebra and* $\mathfrak{g}$ *is a simple Lie algebra from the list* $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$, *but* $\mathfrak{g} \neq \mathfrak{so}_8$. *Then* Γ *is isomorphic to one of graded Lie algebras* $A^1_{(m_1,\dots,m_s);n}$, $A^2_{\mathbf{2}(s);k;m}$, $BD_{\mathbf{2}(s);k;m}$, $C_{\mathbf{2}(s);k;m}$.

Proof. Assume first, that $\mathfrak{g} = \mathfrak{sl}_N$. By proposition 1, it is sufficient to classify maximal diagonalizable subgroups of the group $G = \mathrm{Aut}\,\mathfrak{sl}_N$. The connected component G° of the identity is the group $\mathrm{PGL}(N)$. Suppose, that A is an arbitrary maximal diagonalizable subgroup of the group G. There are two possibilities: either $A \subset G^\circ$ or the subgroup $A_0 = A \cap G$ is a proper subgroup of A.

1) $A \subset G^\circ = \mathrm{PGL}(N)$. It is known, that $\mathrm{PGL}(N) = \mathrm{M}_N$. Therefore, A coincides with the group of diagonal automorphisms of an unrefinable graded associative algebra $\Sigma = (M_N, \Omega)$. It can be easily checked, that in this case $\Gamma = \mathrm{Lie}(\Sigma)/C$. Thus, Γ coincides with a graded Lie algebra from the first series $A^1_{(m_1,\dots,m_s);n}$.

2) Let $A_0 = A \cap G^\circ$ be a proper subgroup of A. Then A_0 is the subgroup of A of index 2 because $G/G^\circ = \mathbb{Z}_2$.

By proposition 4, $A_0 = A(m_1,\dots,m_s) \times A'$, where $N = m_1 \dots m_s n$, $G^\circ \supset \mathrm{PGL}(m_1) \times \dots \times \mathrm{PGL}(m_s) \times \mathrm{PGL}(n)$, $A(m_1,\dots,m_s)$ is the subgroup of $\mathrm{PGL}(m_1) \times \dots \times \mathrm{PGL}(m_s)$, defined in the section 3.5, and A' is a subgroup of a maximal torus of $\mathrm{PGL}(n)$. Let $t \in A \setminus A_0$. Then t is an outer automorphism of the Lie algebra $\mathfrak{sl}_N$. Clearly, t commutes with all elements of A_0. Moreover, the maximality of A leads to a property: every element $g \in G^\circ$, which commutes with t and with all elements of the subgroup A_0, lies in A_0. It is easy to show, that the element t normalize the subgroup K. As a corollary, it can be checked, that this is possible only if $m_1 = \dots = m_s = 2$. Moreover, the cenralizer of A' in the group PGL(n) coincides with a maximal torus $T \subset$ PGL(n). Evidently, t belongs to the normalizer of T. Therefore, t defines the order-two automorphism of T and A' coincides with the set of fixed points of this automorphism. Using the properties of A,which are proven above, it can be easily shown, that the group A coincides with the group of all diagonal automorphisms of a graded Lie algebra from the series $A^2_{\mathbf{2}(s);k;m}$ $(N = 2^{2s}(k+2m))$. Thus, $\Gamma = A^2_{\mathbf{2}(s);k;m}$.

To complete the proof of the theorem, assume that $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$. In any case, we may include the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathfrak{sl}_N$. Denote by $\widehat{G}$ the group of all automorphisms of $\mathfrak{sl}_N$, which preserves the subalgebra $\mathfrak{g}$. It is known, that in any case, except of $\mathfrak{g} = \mathfrak{so}(8)$, there exists an isomorphism $\widehat{G} = \mathbb{Z}_2 \times G$, where $G = \operatorname{Aut}\mathfrak{g}$. The factor $\mathbb{Z}_2$ is generated by an outer automorphism t of $\mathfrak{sl}_N$ and $\mathfrak{g}$ coincides with the set of fixed points of t. It follows immediately from this fact, that a maximal diagonalizable subgroup A of G, together with the element t generates the maximal diagonalizable subgroup $\widehat{A}$ of the group $\operatorname{Aut}\mathfrak{sl}_N$. Hence, $\widehat{A}$ coincides with a group of all diagonal automorphisms of an unrefinable graded Lie algebra $\Gamma' = (\mathfrak{sl}_N, \Omega')$ from the second series. We already have described all such groups. Thus, in order to find all possible subgroups A it is enough to choose index-two subgroups of all possible $\widehat{A}$ with certain (easily stated) properties. It can be proven, that just the subgroups of all fixed points of the automorphisms t_1 and t_2 from the lemma 14 are subgroups of all diagonal automorphisms of unrefinable gradings of Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. This statement gives the classification in these cases. □

6 Special unrefinable graded Lie algebras

In this section we classify unrefinable gradings of Lie algebras G_2, F_4, E_6 and D_4.

Let us briefly describe the main point in the classification of unrefinable gradings of a Lie algebra $\mathfrak{g}$ of a special type. Due to the proposition 1, it is sufficient to classify maximal diagonalizable subgroups of the group $G = \operatorname{Aut}\mathfrak{g}$ up to conjugation in G. We have classified these subgroups using the inductive arguments. Every diagonalizable subgroup A of G is contained in a reductive subgroup K of the maximal rang of the group G. As K, one can choose the centralizer of any nontrivial element $a \in A$, which belongs to the identity component G°. Moreover, if $A \subset G^\circ$, then one can choose K being connected subgroup. Clearly, A is a maximal diagonalizable subgroup of K. The classification of the reductive subgroups of the maximal rank is well-known (it was done by Dynkin [5] for the connected subgroups). Thus, we can reduce the problem to the one for smaller groups. Several technical problems appears on this way, and additional arguments were used to solve them.

6.1 Maximal diagonalizable subgroups of the group $G = \operatorname{Aut} G_2$

It is known, that the group G is a connected simply connected Lie group.

Denote by A_1 a maximal torus of G and by A_2 a Jordan subgroup of G. The subgroup A_2 can be defined as follows (see [2], [3]). The group G contains the connected subgroup K of the type $2A_1$. It can be easily shown that $K = (\mathrm{SL}(2) \times \mathrm{SL}(2))/C_1$, where $C_1 = \{(\lambda E, \lambda E) | \lambda = \pm 1\}$ is the order two subgroup of the center of the group SL(2) × SL(2). There exists a unique (up to conjugation) maximal diagonalizable subgroup $\mathbb{Z}_2^3$ of the group K. This subgroup, considered as a subgroup of G_2 is called Jordan subgroup.

Theorem 4 *Let A be a maximal diagonalizable subgroup of the group G_2. Then A is conjugated either to* (1) *the maximal torus $A_1 = T^2$ or to* (2) *the Jordan subgroup $A_2 = \mathbb{Z}_2^3$. A_1 and A_2 are maximal diagonalizable subgroups of G_2.*

6.2 Maximal diagonalizable subgroups of the group $G = \operatorname{Aut} F_4$

It is known, that the group G is a connected simply connected Lie group. Let us formulate several facts about the definite subgroups of the group G.

Lemma 15 *Let $G = F_4$. Then*
(1) *there exists a unique , up to conjugation, connected subgroup K_1 of the type $2A_2$;*
(2) *$K_1 = (\mathrm{SL}(3) \times \mathrm{SL}(3))/C_1$ where $C_1 = \{(\lambda E, \lambda E) \mid \lambda \in \mathbb{C}^*, \lambda^3 = 1\}$;*
(3) *there exists a unique , up to conjugation, connected subgroup K_2 of the type $A_1 + C_3$;*
(4) *$K_2 = (\mathrm{SL}(2) \times \mathrm{Sp}(6))/C_2$ where $C_2 = \{(\lambda E, \lambda E) \mid \lambda = \pm 1\}$.*

Denote by K_3 the subgroup of the group K_2, which is isomorphic to $(\mathrm{SL}(2) \times (\mathrm{Sp}(2) \otimes \mathrm{SO}(3)))/C_2$, where the tensor product of the linear groups $\mathrm{Sp}(2) \otimes \mathrm{SO}(3)$ is the subgroup of the factor $\mathrm{Sp}(6)$.

Lemma 16 *The subgroup K_3 is isomorphic to the group*
$(\mathrm{SL}(2) \times \mathrm{SL}(2))/\{(\lambda E, \lambda E)|\lambda = \pm 1\} \times \mathrm{SO}(3)$.

The classification for F_4 is given in the theorem:

Theorem 5 *Let A be a maximal diagonalizable subgroup of the group F_4. Then A is conjugated to a subgroup from the list:*
(1) *the maximal torus $A_1 = T^4$;*
(2) *the maximal diagonalizable subgroup $A_2 = \mathbb{Z}_2^3 \times T^1$ of K_3;*
(3) *the maximal diagonalizable subgroup $A_3 = \mathbb{Z}_2^5$ of K_3;*
(4) *the maximal diagonalizable subgroup $A_4 = \mathbb{Z}_3^3$ of K_1.*
A_1, A_2, A_3, A_4 are maximal diagonalizable subgroups of the group F_4.

6.3 Maximal diagonalizable subgroups of the group $G = \mathrm{Aut}\, D_4$

It is known, that the group G can be factorized into the semidirect product of the symmetric subgroup S_3 and the connected normal subgroup $G^{\circ} = \mathrm{SO}(8)/\mathbb{Z}_2$.

Lemma 17 *Let $G = \mathrm{Aut}\, D_4$. Then*
(1) *there exists a unique , up to conjugation, closed subgroup K_1 such that $K_1 = \mathbb{Z}_3 \times \mathrm{PGL}(3)$ and the factor $\mathbb{Z}_3$ is generated by an external automorphism of D_4;*
(2) *there exists a unique , up to conjugation, connected subgroup K_2 such that $K_2 = \mathbb{Z}_3 \times G_2$ and the factor $\mathbb{Z}_3$ is generated by an external automorphism of D_4.*

The classification for the case D_4 is given in the theorem:

Theorem 6 *Let A be a maximal diagonalizable subgroup A of the group $G = \mathrm{Aut}\, D_4$. Then either A defines a classical grading of the Lie algebra $\mathfrak{g} = D_4$ or A is conjugated to a subgroup from the list:*
(1) *the maximal diagonalizable subgroup $A_1 = \mathbb{Z}_3 \times T^2$ of K_1 ;*
(2) *the maximal diagonalizable subgroup $A_2 = \mathbb{Z}_3^3$ of K_1;*
(3) *the maximal diagonalizable subgroup $A_3 = \mathbb{Z}_3 \times \mathbb{Z}_2^3$ of K_2.*
A_1, A_2, A_3 are maximal diagonalizable subgroups of the group $\mathrm{Aut}\, D_4$.

6.4 Maximal diagonalizable subgroups of the group $G = \mathrm{Aut}\, E_6$

It is known, that the group G can be factorized into the semidirect product of a subgroup $\mathbb{Z}_2$ and the identity component G°, which is the quotient group of the connected simply connected group E_6 by its center $C = \mathbb{Z}_3$.

Lemma 18 *Let $G = \mathrm{Aut}\, E_6$. Then*
(1) *there exists a unique , up to conjugation, connected subgroup K_1 of the type $3A_2$;*
(2) *$K_1 = (\mathrm{SL}(3) \times \mathrm{SL}(3) \times \mathrm{SL}(3))/C_1$ where $C_1 = \{(\lambda_1 E, \lambda_2 E, \lambda_3 E) \mid \lambda_i^3 = 1,\ \lambda_1\lambda_2\lambda_3 = 1\}$;*
(3) *there exists a unique , up to conjugation, connected subgroup K_2 of the type $A_1 + A_5$;*
(4) *$K_2 = (\mathrm{SL}(2) \times (\mathrm{SL}(6)/\mathbb{Z}_3))/C_2$ where $C_2 = \{(\lambda E, \lambda E) \mid \lambda = \pm 1\}$;*
(5) *there exists a unique , up to conjugation, connected subgroup K_3 such that $K_3 = \mathbb{Z}_2 \times F_4$ and the factor $\mathbb{Z}_2$ is generated by an external automorphism of E_6;*
(5) *there exists a unique , up to conjugation, connected subgroup K_4 such that $K_4 = \mathbb{Z}_2 \times (\mathrm{Sp}(8)/\mathbb{Z}_2)$ and the factor $\mathbb{Z}_2$ is generated by an external automorphism of E_6.*

Denote by K_5 the subgroup of the group K_2, which is isomorphic to the group $(\mathrm{SL}(2) \times (\mathrm{SL}(2) \otimes \mathrm{SL}(3))/\mathbb{Z}_3)/C_2$, where the factor-group of the tensor product of the linear groups $(\mathrm{SL}(2) \otimes \mathrm{SL}(3))/\mathbb{Z}_3$ is a subgroup of the factor $\mathrm{SL}(6)/\mathbb{Z}_3$ of K_2.

Lemma 19 *The subgroup K_5 is isomorphic to the group*
$((\mathrm{SL}(2) \times \mathrm{SL}(2))/\{(\lambda E, \lambda E)|\lambda = \pm 1\}) \times (\mathrm{SL}(3)/\mathbb{Z}_3)$.

The classification for E_6 is given in the theorem:

Theorem 7 *Let A be a maximal diagonalizable subgroup of the group $G = \mathrm{Aut}\, E_6$. Then A is conjugated to a subgroup from the list:*
(1) *the maximal torus $A_1 = T^6$;*
(2) *the maximal diagonalizable subgroup $A_2 = \mathbb{Z}_3^2 \times T^2$ of K_5;*
(3) *the maximal diagonalizable subgroup $A_3 = \mathbb{Z}_2^3 \times T^2$ of K_5;*
(4) *the maximal diagonalizable subgroup $A_4 = \mathbb{Z}_3^2 \times \mathbb{Z}_2^3$ of K_5;*
(5) *the maximal diagonalizable subgroup $A_5 = \mathbb{Z}_3^4$ of K_1;*
(6) *the maximal diagonalizable subgroup $A_6 = \mathbb{Z}_2 \times T^4$ of K_3;*
(7) *the maximal diagonalizable subgroup $A_7 = \mathbb{Z}_2^4 \times T^1$ of K_3;*
(8) *the maximal diagonalizable subgroup $A_8 = \mathbb{Z}_2^6$ of K_3;*
(9) *the maximal diagonalizable subgroup $A_9 = \mathbb{Z}_2 \times \mathbb{Z}_3^3$ of K_3;*
(10) *the maximal diagonalizable subgroup $A_{10} = \mathbb{Z}_2^3 \times T^2$ of K_4;*
(11) *the maximal diagonalizable subgroup $A_{11} = \mathbb{Z}_2^5 \times T^1$ of K_4;*
(12) *the maximal diagonalizable subgroup $A_{12} = \mathbb{Z}_2^7$ of K_4.*
The subgroups $A_1, \ldots, A_{12}$ are maximal diagonalizable subgroups of the group $\mathrm{Aut}\, E_6$.

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