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Hilbert flag varieties and associated representations [1]

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Let H be a complex Hilbert space. If H is finite dimensional, then one knows from the Borel-Weil theorem that the finite dimensional irreducible representations of the general linear group $\mathrm{GL}(H)$ can be realized geometrically as the natural action of the group $\mathrm{GL}(H)$ on the space of global holomorphic sections of a holomorphic line bundle over a space of flags in H. By choosing a basis of H, one can identify this space of holomorphic sections with a space of holomorphic functions on $\mathrm{GL}(H)$ that are certain polynomial expressions in minors of the matrices corresponding to the elements of $\mathrm{GL}(H)$.

In this overview we will give an infinite dimensional analogue of all these representations. Thereto we take a separable Hilbert space H. In H we consider a collection of flags that can be given a Hilbert space structure. It is a homogeneous space for an analogue of the general linear group, the so-called restricted linear group. There is a proper analogue of the notion of maximal torus in this restricted linear group. Over the flag variety there exist line bundles that are similar to the finite dimensional ones. In the "dominant" case the space of global holomorphic sections of such a line bundle turns out to be non-trivial. However, the action of the restricted linear group can, in general, not be lifted to the line bundle under consideration and one has to pass to a central extension of this group. The representation space contains a unique section on which the maximal torus acts by the dominant weight. It is the generator of an irreducible highest weight module of the central extension.

Keywords: Hilbert flag varieties, holomorphic line bundles, global sections, central extension, maximal torus, dominant weight, highest weight module

§ 1. The flag variety

Let H be a separable complex Hilbert space with innerproduct $<\cdot,\cdot>$. One will consider certain finite chains of subspaces in H and they will be called flags as in the finite dimensional case. First one has to specify the "size" of the components of the flag. Therefore one starts with an orthogonal decomposition of H,

$$H = H_1 \oplus \ldots \oplus H_m, \quad \text{where } H_i \perp H_j \text{ for } i \neq j. \tag{1.1}$$

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One assumes that $m_i = \dim H_i$ satisfies $1 \leqslant m_i \leqslant \infty$ and that H has a Hilbert basis indexed by the integers.

Remark 1. A natural way to get such a decomposition is to consider in $\mathrm{GL}(H)$ the maximal torus $T(\mathcal{N})$ of all invertible diagonal operators that differ from the identity by a nuclear operator. Concretely, it consists of all operators of the form

$$\mathrm{diag}(\{1+t_s\}), \text{ with } 1+t_s \neq 0 \text{ and } \sum_{s\in\mathbb{Z}} |t_s| < \infty.$$

In $T(\mathcal{N})$ we have the dense subgroup T_f given by

$$T_f = \{t \mid t = \mathrm{diag}(\{1+t_s\}) \in T(\mathcal{N}),\ t_s \neq 0 \text{ for only finitely many } s \text{ in } \mathbb{Z}\}$$

Any analytic grouphomomorphism of T_f into $\mathbb{C}^*$ has the form

$$t = \mathrm{diag}((1+t_s)) \longmapsto \prod_{s\in\mathbb{Z}} (1+t_s)^{m_s} = \chi_{\underline{m}}(t),$$

where $\underline{m} = \{m_s\}$, with $m_s \in \mathbb{Z}$ for all $s \in \mathbb{Z}$. This character $\chi_{\underline{m}}$ can be continued to an analytic character of $T(\mathcal{N})$ if and only if there are only finitely many different m_s, $s \in \mathbb{Z}$. This extension of $\chi_{\underline{m}}$ is also denoted by $\chi_{\underline{m}}$ and one writes $\widehat{T}$ for the group of analytic characters of $T(\mathcal{N})$. To each $\chi_{\underline{m}}$ one associates the decomposition corresponding to the spans of the directions with the same value of m_s.

Let p_i, $1 \leqslant i \leqslant m$, be the orthogonal projection of H onto H_i. Then we will use throughout this paper the following

Notation 1. If g belongs to $\mathcal{B}(H)$, the space of bounded linear operators from H to H, then $g = (g_{ij})$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$, denotes the decomposition of g with respect to the $\{H_i \mid 1 \leqslant i \leqslant m\}$. That is to say $g_{ij} = p_i \circ g \mid H_j$.

To the decomposition (1) one associates the basic flag $F^{(0)}$ given by

$$0 \subset H_1 \subset \ldots \subset \bigoplus_{j=1}^{r} H_j \subset \ldots \subset H.$$

In H one considers flags $F = \{F(0), \ldots, F(m)\}$, that is to say chains of closed subspaces of H,

$$\{0\} = F(0) \subset F(1) \subset \ldots \subset F(m) = H,$$

that are of the same "size" as the basic flag $F^{(0)}$, i.e. for all i, $1 \leqslant i \leqslant m$,

$$\dim(F(i)/F(i-1)) = \dim H_i\,.$$

To such a flag F is associated an orthogonal decomposition of H,

$$H = F_1 \oplus \ldots \oplus F_m\,, \quad \text{where } F_i = F(i) \cap F(i-1)^{\perp}.$$

One will denote such a flag F by $F = \{F(0), \ldots, F(m)\}$ as well as $F = \{F_1, \ldots, F_m\}$.

The class of flags one obtains in this way is still too wide and it will be required that our flags do not differ too much from the basic flag.

Definition 1. Let $\mathfrak{F}$ be the collection of flags $F = \{F_1, \ldots, F_m\}$, satisfying $\dim F_i = \dim H_i$, and for all i and j with $j \neq i$, the orthogonal projection $p_j : F_i \to H_j$ is a Hilbert-Schmidt operator. One calls $\mathfrak{F}$ the *flag variety* corresponding to the decomposition (1).

As in the finite dimensional case, the space $\mathfrak{F}$ is a homogeneous space for a certain unitary group. Let $F = \{F_1, \ldots, F_m\}$ belong to $\mathfrak{F}$. From the definition of $\mathfrak{F}$ one knows that there is for each i, $1 \leqslant i \leqslant m$, an isometry u_i between H_i and F_i. If one puts $u = u_1 \oplus \ldots \oplus u_m$, then u belongs to the group of unitary transformations, $U(H)$, of H and for each i, $1 \leqslant i \leqslant m$, one has

$$u\left(\bigoplus_{j=1}^{i} H_j\right) = \bigoplus_{j=1}^{i} F_j.$$

In other words, the flag F is the image under u of the basic flag. The condition defining $\mathfrak{F}$ implies that $u = (u_{ij})$ satisfies: u_{ij} is a Hilbert-Schmidt operator for $i \neq j$. This brings us to the introduction of the following group.

Definition 2. The *restricted unitary group*, $U_{\mathrm{res}}(H)$, consists of all $u = (u_{ij})$ in $U(H)$ such that u_{ij} is a Hilbert-Schmidt operator if $i \neq j$.

Clearly the stabilizer of $F^{(0)}$ in $U_{\mathrm{res}}(H)$ is equal to $\prod\limits_{i=1}^{m} U(H_i)$ and therefore one can identify $\mathfrak{F}$ with the homogeneous space

$$U_{\mathrm{res}}(H) \Big/ \prod_{i=1}^{m} U(H_i).$$

For several reasons, like the description of the manifold structure on $\mathfrak{F}$ and the consideration of non-unitary flows on $\mathfrak{F}$, it is convenient to have a description of $\mathfrak{F}$ as the homogeneous space of a larger group of automorphisms of H. The Banach structure of this group follows directly from that of its Lie algebra. Therefore one starts with the analogue of the Lie algebra of the general linear group.

Definition 3. A *restricted endomorphism* of H is a $u = (u_{ij})$ in $\mathcal{B}(H)$ such that u_{ij} is a Hilbert-Schmidt operator for all $i \neq j$. One denotes the space of all restricted endomorphisms of H by $\mathcal{B}_{\mathrm{res}}(H)$.

The space $\mathcal{B}_{\mathrm{res}}(H)$ is a subalgebra of $\mathcal{B}(H)$ since the collection of Hilbert-Schmidt operators is closed under left and right multiplication with bounded operators. Hence it is also a Lie subalgebra of the Lie algebra $\mathcal{B}(H)$. On $\mathcal{B}_{\mathrm{res}}(H)$ one will introduce a norm. The algebra $\mathcal{B}_{\mathrm{res}}(H)$ becomes a Banach algebra if one equips it with the norm

$\|\cdot\|_2$ defined by

$$\|u\|_2 = \|u\| + \sum_{i\neq j} \|u_{ij}\|_{\mathcal{HS}}.$$

Since the adjoint of a Hilbert-Schmidt operator is again Hilbert-Schmidt, it is clear that $\mathcal{B}_{\rm res}(H)$ is stable under "taking adjoints". If $\mathrm{GL}(H)$ denotes the group of invertible elements in $\mathcal{B}(H)$, then one considers

Definition 4. The *restricted linear group* $\mathrm{GL}_{\rm res}(H)$ consists of g such that g belongs to $\mathrm{GL}(H)\cap\mathcal{B}_{\rm res}(H)$.

One easily verifies that $\mathrm{GL}_{\rm res}(H)$ consists of the invertible elements in $\mathcal{B}_{\rm res}(H)$. Thus one can identify the tangent space at any point of $\mathrm{GL}_{\rm res}(H)$ with $\mathcal{B}_{\rm res}(H)$.

With each g in $\mathrm{GL}_{\rm res}(H)$ one can associate the flag

$$0\subset gH_1\subset g(H_1\oplus H_2)\subset\ldots g(H_1\oplus\ldots\oplus H_i)\subset\ldots\subset H.$$

From the definition of $\mathrm{GL}_{\rm res}(H)$ one sees directly that this flag belongs to $\mathfrak{F}$. The stabilizer in $\mathrm{GL}_{\rm res}(H)$ of the basic flag is the "parabolic subgroup" P consisting of upper triangular matrices $g\in\mathrm{GL}_{\rm res}(H)$:

$$g=\begin{pmatrix} g_{11} & \cdots & \cdots & g_{1m}\\ 0 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \ldots & 0 & g_{mm}\end{pmatrix},$$

with $g_{ii}\in\mathrm{GL}_{\rm res}(H_i)$, $1\leqslant i\leqslant m$. Thus one can identify $\mathfrak{F}$ also with the homogeneous space $\mathrm{GL}_{\rm res}(H)/P$. Let $\tau:\mathrm{GL}_{\rm res}(H)\to\mathfrak{F}$ be the projection $\tau(g)=g\cdot F^{(0)}$. On $\mathfrak{F}$ one puts the quotient topology that makes τ into an open continuous map.

To get an idea how the space $\mathfrak{F}$ locally looks like near the basic flag, one considers the set Ω in $\mathrm{GL}_{\rm res}(H)$ consisting of g such that matrices

$$\begin{pmatrix} g_{11} & \cdots & g_{1i}\\ \vdots & & \vdots\\ g_{i1} & \cdots & g_{ii}\end{pmatrix}$$

belong to $\mathrm{GL}_{\rm res}(H_1\oplus\ldots\oplus H_i)$ for all $i\leqslant m$. The set Ω is open and, as in the finite dimensional case, can be decomposed. For, let U_- be the subgroup of $\mathrm{GL}_{\rm res}(H)$ consisting of $g=(g_{ij})$ such that $g_{ii}=\mathrm{Id}_{H_i}$ for all i and $g_{ij}=0$ for $j>i$. The group U_- is given the topology induced by that of $\mathrm{GL}_{\rm res}(H)$. The set Ω has a similar description as in the finite dimensional situation:

Lemma 1 *The map $(u,p)\mapsto up$ from $U_-\times P\to\mathrm{GL}_{\rm res}(H)$ determines a homeomorphism between $U_-\times P$ and Ω.*

An easy conseqence is that the flagvariety $\mathfrak{F}$ is a Hilbert manifold based on the Lie algebra E of U_-.

§ 2. The connected components of $\mathfrak{F}$

Let $g = (g_{ij})$ be an element of $\mathrm{GL}_{\mathrm{res}}(H)$ and put $g^{-1} = (h_{ij})$. Then one has, by definition for all i, $1 \leqslant i \leqslant m$,

$$g_{ii}h_{ii} = \mathrm{Id}_{H_i} - \sum_{k \neq i} g_{ik}h_{ki}.$$

This implies that each g_{ii} is a Fredholm operator, that is to say it has a finite dimensional kernel and cokernel. The collection of Fredholm operators on a Hilbert space K is denoted by $\Phi(K)$ and it is an open part of the space $\mathcal{B}(K)$. Its connected components are given by the index, which is defined as

$$\mathrm{ind}(B) = \dim\ker(B) - \dim\mathrm{coker}(B), \quad \text{for } B \in \Phi(K).$$

Since all off-diagonal operators are Hilbert-Schmidt and hence compact, the operator

$$\widetilde{g} = \begin{pmatrix} g_{11} & & 0 \\ & \ddots & \\ 0 & & g_{mm} \end{pmatrix},$$

where $g = (g_{ij}) \in \mathrm{GL}_{\mathrm{res}}(H)$, is a Fredholm operator of index zero. Hence the indices of the g_{ii}, $1 \leqslant i \leqslant m$, satisfy

$$\sum_{i=1}^{m} \mathrm{ind}(g_{ii}) = 0 \quad \text{and} \quad \mathrm{ind}(g_{kk}) = 0 \quad \text{if} \quad m_k < \infty.$$

These relations lead to the introduction of the subgroup Z of $\mathbb{Z}^m$ consisting of vectors $z = (z_1, \ldots, z_m)$, $z_i \in \mathbb{Z}$, such that

$$\sum_{i=1}^{m} z_i = 0, \quad \text{and} \quad z_k = 0 \quad \text{if} \quad m_k < \infty.$$

The standard properties of the index imply that the map $i : \mathrm{GL}_{\mathrm{res}}(H) \to Z$,

$$g \mapsto (\mathrm{ind}(g_{11}), \ldots, \mathrm{ind}(g_{mm})),$$

is a continuous grouphomomorphism. Hence the subsets $\mathrm{GL}_{\mathrm{res}}^{(z)}(H)$ of $\mathrm{GL}_{\mathrm{res}}(H)$, consisting of g such that $i(g) = z$ with $z \in Z$ are open. In fact, they are exactly the connected components of $\mathrm{GL}_{\mathrm{res}}(H)$, for one can show

Proposition 1 *For each $z \in Z$, the set $\mathrm{GL}_{\mathrm{res}}^{(z)}(H)$ is non-empty and connected.*

Since the parabolic group P is connected, one concludes for the flag variety

Corollary 3 *The connected components of* $\mathfrak{F}$ *are given by*

$$\mathfrak{F}^{(z)} = \left\{ g.F^{(0)} \mid g \in \mathrm{GL}_{\mathrm{res}}^{(z)}(H) \right\}.$$

Remark 2. A holomorphic line bundle L over $\mathfrak{F}$ consists simply of a collection of holomorphic line bundles $\{L_z \to \mathfrak{F}^{(z)} \mid z \in Z\}$. Therefore one restricts one's attention to holomorphic line bundles over $\mathfrak{F}^{(0)}$ in the next section.

§ 3. The holomorphic line bundles over $\mathfrak{F}^{(0)}$

Each F in $\mathfrak{F}^{(0)}$ is equal to $g.F^{(0)}$ with $g \in \mathrm{GL}_{\mathrm{res}}(H)$ of the form

(a) For each i, $1 \leqslant i \leqslant m$, $g_{ii} = \mathrm{Id}_{H_i} +$ a “finite-size” operator.

(b) For all i and j, $i < j$, g_{ij} is a “finite-size” operator.

(c) For all i and j, $j < i$, g_{ij} belongs to $\mathcal{HS}(H_j, H_i)$.

Note that for all the operators g_{ii} from (a) one can speak of $\det(g_{ii})$. Since one is working in an analytic setting it is convenient to consider a somewhat wider class of operators such that on one hand the framework is complete and on the other one can take determinants of certain minors. Recall that the determinant is defined for each operator of the form “identity + *a* nuclear operator”. Therefore one introduces $B_2(H)$ consisting of $g \in \mathcal{B}_{\mathrm{res}}(H)$ such that $g_{ii} - \mathrm{Id}_{H_i} \in \mathcal{N}(H_i)$ and $g_{ij} \in \mathcal{HS}(H_j, H_i)$ for $i \neq j$. On $B_2(H)$ one puts a different topology than the one induced by $\mathcal{B}_{\mathrm{res}}(H)$. For, let $\mathcal{Z}$ be the subspace of $\mathcal{B}_{\mathrm{res}}(H)$ consisting of $b \in \mathcal{B}_{\mathrm{res}}(H)$ such that $b_{ii} \in \mathcal{N}(H_i)$ and $b_{ij} \in \mathcal{HS}(H_j, H_i)$ for $i \neq j$. Then $\mathcal{Z}$ is a Banach space if it is equipped it with the norm $\|\cdot\|_{\mathcal{Z}}$ given by

$$\|b\|_{\mathcal{Z}} = \sum_{i \neq j} \|b_{ij}\|_{\mathcal{HS}} + \sum_{i=1}^{m} \|b_{ii}\|_{\mathrm{tr}}.$$

The collection $B_2(H)$ is nothing but $\mathcal{Z}$ shifted by the identity and one transfers the Banach structure on $\mathcal{Z}$ to $B_2(H)$ by means of the map $g \mapsto g + \mathrm{Id}$. Since the product of two Hilbert-Schmidt operators is nuclear, one sees that $B_2(H)$ is closed under multiplication. Moreover the multiplication with an element of $B_2(H)$ is an analytic map from $B_2(H)$ to itself. In $B_2(H)$ one has the subgroup U_- and its “adjoint” the group

$$U_+ = \{u^* \mid u \in U_-\}.$$

Consider an element b in $B_2(H)$. Since b_{ii} has the form "$\mathrm{Id}_{H_i}+$ a nuclear operator", one can find an operator $\widetilde{b}_{ii}$ in $\mathrm{GL}(H_i)$ such that $\widetilde{b}_{ii}-\mathrm{Id}_{H_i}$ and $(\widetilde{b}_{ii})^{-1}-\mathrm{Id}_{H_i}$ both belong to $\mathcal{N}(H_i)$. Now one defines $u=(u_{ij})$ in U_- and $v=(v_{ij})$ in U_+ by

$$u_{ii}=v_{ii}=\mathrm{Id}_{H_i}, u_{ij}=-b_{ij}(\widetilde{b}_{jj})^{-1} \quad \text{if } i>j \text{ and } u_{ij}=0 \text{ if } j>i,$$
$$v_{ij}=-(\widetilde{b}_{ii})^{-1}b_{ij} \quad \text{if } i<j \text{ and } v_{ij}=0 \text{ if } i>j.$$

A direct verification shows that ubv belongs to $\mathrm{Id}+\mathcal{N}(H)$. Since $B_2(H)$ is closed with respect to taking adjoints, there holds

Lemma 2 *Every $b\in B_2(H)$ can be written in the form $b=u_1b_1v_1$ or $b=v_2b_2u_2$, where u_1 and u_2 belong to U_-, v_2 and v_1 belong to U_+ and b_1 and b_2 lie in* $\mathrm{Id}+\mathcal{N}(H)$.

If one takes into account that for each i, the operator $\widetilde{b}_{ii}$ can be chosen of the form $b_{ii}+f_{ii}$, where f_{ii} is finite dimensional, then for all c in $B_2(H)$ sufficiently close to b one can take $\widetilde{c}_{ii}=c_{ii}+f_{ii}$. By using this, one shows easily that the map $\det: B_2(H)\to\mathbb{C}$, defined by $\det(b)=\det(u_1b_1v_1)=\det(b_1)$, where $b=u_1b_1v_1$ as in the lemma above, is well-defined and analytic on $B_2(H)$.

Remark 3. Since the operators in $\mathrm{Id}+\mathcal{N}(H)$ lie dense in $B_2(H)$ and since det is multiplicative on $\mathrm{Id}+\mathcal{N}(H)$, one gets that for each b_1 and b_2 in $B_2(H)$

$$\det(b_1b_2)=\det(b_1)\det(b_2)$$

From the fact that an operator g of the form $\mathrm{Id}+\mathcal{N}(H)$ is invertible if and only if $\det(g)$ is non-zero, one sees that the invertible elements of $B_2(H)$ form a group $\mathcal{G}$ consisting of $b\in B_2(H)$ with $\det(b)\neq 0$. Clearly $\mathcal{G}$ is a Banach Lie group with Lie algebra $\mathcal{Z}$ and it acts analytically and transitively on $\mathfrak{F}^{(0)}$. The stabilizer $\mathcal{T}$ of $F^{(0)}$ in $\mathcal{G}$ consists of upper triangular matrices

$$t=\begin{pmatrix} t_{11} & \dots & & t_{1m} \\ 0 & & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & t_{mm} \end{pmatrix}$$

with $t_{ii}\in\{\mathrm{Id}+\mathcal{N}(H_i)\}\cap\mathrm{GL}(H_i)$ and $t_{ij}\in\mathcal{HS}(H_j,H_i)$ for $j>i$. Thus one can identity $\mathfrak{F}^{(0)}$ with the homogeneous space $\mathcal{G}/\mathcal{T}$.

For each $\underline{k}=(k_1,\dots,k_m)$ in $\mathbb{Z}^m$, one defines $\psi_{\underline{k}}$ in $\widehat{T}$ by

$$\psi_{\underline{k}}=(\mathrm{diag}\{1+t_s\})=\prod_{s_1\in S_1}(1+t_{s_1})^{k_1}\prod_{s_2\in S_2}(1+t_{s_2})^{k_2}\dots\prod_{s_m\in S_m}(1+t_{s_m})^{k_m}\ .$$

Clearly $\psi_{\underline{k}}$ extends to an analytic character of $\mathcal{T}$ by means of the formula

$$\psi_{\underline{k}}(t)=\det(t_{11})^{k_1}\dots\det(t_{mm})^{k_m}.$$

To each $\psi_{\underline{k}}$ one can associate a holomorphic line bundle $L(\underline{k})$ over $\mathfrak{F}^{(0)} = \mathcal{G}/\mathcal{T}$. It is defined as follows: consider on the space $\mathcal{T} \times \mathbb{C}$ the equivalence relation

$$(g_1, \lambda_1) \sim (g_2, \lambda_2) \Leftrightarrow g_1 = g_2 \circ t, \quad \text{with } t \in \mathcal{T} \text{ and } \lambda_2 = \lambda_1 \psi_{\underline{k}}(t).$$

The space $\mathcal{T} \times \mathbb{C}$ modulo this equivalence relation is $L(\underline{k})$. For each $g \in \mathcal{G}$ and each λ in $\mathbb{C}$, one denotes the equivalence class to which the pair (g, λ) belongs by $[g, \lambda]$. There is a natural projection $\pi_{\underline{k}} : L(\underline{k}) \to \mathfrak{F}^{(0)}$ given by

$$\pi_{\underline{k}}([g, \lambda]) = g \cdot F^{(0)}.$$

The space $L(\underline{k})$ is a Hilbert manifold based on the Hilbert space $E \oplus \mathbb{C}$.

§ 4. The central extension

There is a natural analytic action of the group $\mathcal{G}$ on the space $L(\underline{k})$ by left translations

$$g_1 \cdot [g_2, \lambda] = [g_1 g_2, \lambda].$$

This is a lifting of the natural action of $\mathcal{G}$ on $\mathfrak{F}^{(0)}$ to one on $L(\underline{k})$. However, the natural action of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ can, in general, not be lifted to one on $L(\underline{k})$. Such an attempt may lead to nontrivial central extensions of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ as one will show.

Note that each g in $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ can be written as $g = dg_2$, with $g_2 \in \mathcal{G}$ and d belonging to the "diagonal" subgroup D of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ consisting of $g = (g_{ij})$ with $g_{ij} = 0$ if $i \neq j$. Clearly the group D normalizes the group $\mathcal{G}$. Since the determinant of an operator of the form "identity + nuclear" is invariant under conjugation with an invertible operator, one gets that D centralizes each $\psi_{\underline{k}}$, i.e. for each t in $\mathcal{T}$ and each d in D there holds

$$\psi_{\underline{k}}(d\, t\, d^{-1}) = \psi_{\underline{k}}(t).$$

This fact permits you to lift the action of D on $\mathfrak{F}^{(0)}$ to one on $L(\underline{k})$ by means of

$$d \cdot [g,\, \lambda] = [d\, g\, d^{-1},\, \lambda].$$

For an element d from $D \cap \mathcal{G}$, this action differs by a factor $\psi_{\underline{k}}(d^{-1})$ from the action induced by that of $\mathcal{G}$. Hence one cannot combine them to an action of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ on $\mathfrak{F}^{(0)}$. To overcome this problem a group extension G of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ will be built. It is defined by

$$G = \left\{ (g, d) \;\middle|\; g \in \mathrm{GL}^{(0)}_{\mathrm{res}}(H),\, d \in D \quad \text{and} \quad gd^{-1} \in \mathcal{G} \right\}.$$

As one verifies directly this group acts on $L(\underline{k})$ by means of

$$(g, d)[g_1, \lambda_1] = [g g_1 d^{-1}, \lambda_1].$$

It is simply the combination of the $\mathcal{G}$-action and the D-action given above. Let $\pi : G \to \mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ be the canonical projection, i.e. $\pi((g,d)) = g$ for all $(g,d) \in G$. For certain subgroups of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ there exist several ways to embed them into G. Therefore we introduce special notations for two of them. Let $\underline{i}$ resp. $\underline{j}$ be the embedding of $\mathcal{G}$ resp. D into G given by

$$\underline{i}(g) = (g, \mathrm{Id}) \quad \text{and} \quad \underline{j}(d) = (d, d).$$

As a group G is the semi-direct product of $\underline{i}(\mathcal{G})$ and $\underline{j}(D)$. We equip each $\mathrm{GL}(H_i)$ with the operator norm topology and we put on $\underline{j}(D)$ the product Banach Lie group structure. On $\underline{i}(\mathcal{G})$ we take the Banach structure based on $\mathcal{Z}$. The conjugation with an element d of D defines an analytic diffeomorphism of $\mathcal{G}$. Hence if we put on G the product topology of $\underline{i}(\mathcal{G})$ and $\underline{j}(D)$, it becomes a Banach Lie group based on

$$\bigoplus_{i=1}^{m} \mathcal{B}(H_i) \oplus \mathcal{Z}.$$

The group G is a fiber bundle over $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ with fiber $\mathcal{T} \cap D$. Next one tries to minimalize the extension of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ that acts on $\mathfrak{F}^{(0)}$ and $L(\underline{k})$. Thereto one considers the action of the kernel of π on $L(\underline{k})$

$$(\mathrm{Id}, d) \cdot [g, \lambda] = [gd^{-1}, \lambda] = [g, \psi_{\underline{k}}(d^{-1})\lambda].$$

In particular the subgroup $D(\underline{k})$ of G consisting of (Id, d) with $\psi_{\underline{k}}(d) = 1$ acts trivially on $L(\underline{k})$ and one sees that it suffices to consider the extension $G(\underline{k}) = G/D(\underline{k})$ of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$. If the character $\psi_{\underline{k}}$ is trivial, i. e. $\underline{k} = 0$, then $G(\underline{k})$ is just $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$. For $\underline{k} \neq 0$, one computes directly that $G(\underline{k})$ is a central extension of $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ with $\mathrm{Ker}(\pi)/D(\underline{k}) \cong \mathbb{C}^*$.

One can describe such an extension with a Borel 2-cocycle $\alpha : \mathrm{GL}^{(0)}_{\mathrm{res}}(H) \times \mathrm{GL}^{(0)}_{\mathrm{res}}(H) \to \mathbb{C}^*$. It can be constructed as follows: take a section ρ of the fiber bundle $G \xrightarrow{\pi} \mathrm{GL}^{(0)}_{\mathrm{res}}(H)$, i. e. for each g in $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ one has

$$\rho(g) = (g, q(g)) \quad \text{with} \quad q(g) \in D.$$

By definition there holds for each g_1 and g_2 in $\mathrm{GL}^{(0)}_{\mathrm{res}}(H)$ that

$$q(g_1)\, q(g_2)\, q(g_1\, g_2)^{-1} \in D \cap \mathcal{G}.$$

Thus one gets for the action on $L(\underline{k})$ the relation

$$\begin{aligned} \rho(g_1 g_2) \cdot [g, \lambda] &= \rho(g_1) \cdot \left\{\rho(g_2) \cdot [g,\, \lambda\, \psi_{\underline{k}}(q(g_1)\, q(g_2)\, q(g_1\, g_2)^{-1})]\right\} \\ &:= \rho(g_1) \cdot \left\{\rho(g_2) \cdot [g,\, \lambda\, \alpha(g_1,\, g_2)^{-1}]\right\}. \end{aligned}$$

The group $G(\underline{k})$ is then isomorphic as a group to the product space $\mathrm{GL}^{(0)}_{\mathrm{res}}(H) \times \mathbb{C}^*$ with the multiplication

$$(g_1, \lambda_1) * (g_2, \lambda_2) = (g_1\, g_2,\, \lambda_1\, \lambda_2\, \alpha\,(g_1,\, g_2)).$$

A detailed analysis of this central extension yields the following general result:

Theorem 1

(a) *The extension* $G(\underline{k})$ *is always trivial if there is at most one infinite* m_i.

(b) *If there are at least two infinite dimensional components in the basic flag, then* $G(\underline{k})$ *is trivial if and only if for all* i *and* j,

$$m_i = m_j = \infty \Rightarrow k_i = k_j.$$

(c) *If* $k_i \neq k_j$ *for infinite dimensional* H_i *and* H_j*, then the corresponding Lie algebra 2-cocycle for the extension* $G(\underline{k})$ *is given by*

$$d\alpha(X,Y) = \sum_{i=1}^{m} k_i \operatorname{Trace}\left\{\sum_{j\neq i} X_{ij}Y_{ji} - \sum_{j\neq i} Y_{ij}X_{ji}\right\}.$$

§ 5. Holomorphic sections of $L(\underline{k})$

Let $\mathfrak{L}(\underline{k})$ denote the space of global holomorphic sections of $L(\underline{k})$. The space $\mathfrak{L}(\underline{k})$ is given the topology of uniform convergence on compact subsets of $\mathfrak{F}^{(0)}$. It becomes then a complete locally convex space. Let $\underline{f} : \mathfrak{F}^{(0)} \to L(\underline{k})$ belong to $\mathfrak{L}(\underline{k})$ then it can be written as

$$\underline{f}(g \cdot F^{(0)}) = [g, f(g)], \quad \text{for all } g \in \mathcal{G},$$

where $f : \mathcal{G} \to \mathbb{C}$ is a holomorphic function satisfying

$$f(gt) = f(g)\psi_{\underline{k}}(t)^{-1} \quad \text{for all } g \in \mathcal{G} \text{ and all } t \in \mathcal{T}. \tag{5.2}$$

Thus we can identify $\mathfrak{L}(\underline{k})$ with the space of holomorphic functions on $\mathcal{G}$ that satisfy this condition. Since each (g,d) in G acts as an analytic diffeomorphism on as well $\mathfrak{F}^{(0)}$ as $L(\underline{k})$, one gets a natural action of G on $\mathfrak{L}(\underline{k})$ that corresponds on the functions on $\mathcal{G}$ satisfying (2) to

$$(g,d)(f)(g_1) = f(g^{-1}\, g_1\, d), \quad \text{with } g_1 \in \mathcal{G} \text{ and } (g,d) \in G.$$

By approximating the flag variety $\mathfrak{F}^{(0)}$ with finite dimensional flag varieties and using the representation theory in that case, one arrives at the following results:

Theorem 2

(a) *The space* $\mathfrak{L}(\underline{k})$ *is non-zero if and only if* $k_1 \leqslant \ldots \leqslant k_m$.

(b) *If the space* $\mathfrak{L}(\underline{k})$ *is non-zero, then the space of vectors in* $\mathfrak{L}(\underline{k})$ *on which* $T(\mathcal{N})$ *acts by* $\psi_{\underline{k}}$ *is one-dimensional.*

(c) *Let* v *be a nonzero in* $\mathfrak{L}(\underline{k})$ *on which* $T(\mathcal{N})$ *acts by* $\psi_{\underline{k}}$*, then it is the generator of an irreducible highest weight module of* $G(\underline{k})$.