On unitary classification of weakly centered operators.

Alexandra Piryatinskaya
Inst. Econom. Manag. Publ. Law., 30-32 Legendarna, 252040 Kiev, Ukraine

Introduction.

This paper is devoted to study up to unitary equivalence of some classes of non-selfadjoint operators acting in a Hilbert space.

Let N be a normal operator, then the spectral theorem gives a description of N up to unitary equivalence. One can also use the weaker condition $[WW^*, W] = 0$. Such operators are called quasinormal [2, 9]. They are described up to unitary equivalence in [2].

Another class of operators which satisfy the condition $[XX^*, X^*X] = 0$ were studed in [3, 4, 19] and other. They are called binormal in [3, 4]. In [19] they are called weakly centered. We will follow this last terminology. The operators X such that the family $\{X^i(X^*)^i, (X^*)^j X^j\}_{i,j\in\mathbb{N}}$ consists of mutually commuting operators were studed in [17, 19] and other. The following [14]), we will show that the problem to describe the weakly centered operators up to unitary equivalence is $*$-wild. For centered operators, which are non-type I followig [14], we will show that the problem of unitary classification are not $*$-wild (see also [18]).

For studying of classes of non-selfadjoint operators we use the framework of representation theory (see Sec.1). In Sec. 2 we developing [11]-[14] and following [15, 16] explaine an ideology of $*$-wildness. Some properties of $*$-wild of $*$-algebras are also considered. In Sec. 3.1 we will show that the problem of describing of weakly centered operators (up to unitary equivalence) is $*$-wild. The same holds even if the operators are a partial isometries. In Sec. 3.2 we will show that centered operators are not $*$-wild.

1 Non-selfadjoint operators and representation of $*$-algebras.

Let $\mathcal{X}$ and $\mathcal{X}^*$ are operators in a Hilbert space $\mathcal{H}$ which satisfy the polynomial relations $P_j(\mathcal{X}, \mathcal{X}^*) = 0$, $j = 1, \ldots, m$. Then one can consider a $*$-algebra $\mathfrak{A} = \mathbb{C}\langle x, x^* \mid P_j(x, x^*) = 0, j = 1, \ldots, m\rangle$ generating by letters x, x^* which satisfy relations $P_j(x, x^*) = 0$, $j = 1, \ldots, m$.

A representation of a $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism $\pi : \mathfrak{A} \to L(H)$ into the algebra $L(H)$ of bounded operators in a complex separable Hilbert space H. Each representation π of the $*$-algebra $\mathfrak{A}$ determines the bounded operators $\pi(x) = X$, $\pi(x^*) = X^*$, such that

$$P(X, X^*) = 0 \qquad j = 1, \ldots, m. \tag{1}$$

Conversly, a given operators X and X^* such that $P_j(X, X^*) = 0$, $j = 1, \ldots, m$ uniquely define a representation of the whole algebrs $\mathfrak{A}$. Thus the problem of unitary description of operators X and X^*, satisfying relations (1) is a problem of description, up to unitary equivalence, of representation of the $*$-algebra $\mathfrak{A}$. In the sequal, we will be considering the unitary classification problems for representations of the following $*$-algebras (and, correpondingly, the unitary classification problems for following classes of non-selfadjoint operators):

1)$\mathfrak{W} = \mathbb{C}\langle x, x^* \mid [xx^*, x^*x] = 0\rangle$ (classification problem for weakly centered operators);

2) $\mathfrak{N} = \mathbb{C}\langle y, y^* \mid (y^*y)^2 = y^*y\rangle$ (classification problem for partial isometries);

3) $\mathfrak{M} = \mathbb{C}\langle x, x^* \mid [xx^*, x^*x] = 0, (x^*x)^2 = x^*x\rangle$ (classification problem for weakly centered operators which are partial isametries).

4)$\mathfrak{C} = \mathbb{C}\langle x, x^* \mid \forall i, j[x^i, (x^*)^i, x^j(x^*)^j] = [x^i, (x^*)^i, (x^*)^j, x^j] =$
$= [(x^*)^i, x^i, (x^*)^j x^j] = 0\rangle$ (classification problem of centered operators).

The notions of unitary equivalence, irreducible representations and other terminology are as usual in representation theory.

2 On $*$-wild $*$-algebras.

In this part we explaine a definition of $*$-wildness for $*$-algebras, following [15, 16]. and give some properties of these algebras. In the theory of representations of algebras, it was suggested [7] to consider a representation problem to be wild if it contains a standard difficult problem of the representation theory, e.g. the problem to describe, up to similyarity, a pair of matrices without relations. To define an analogue of wildness for $*$-algebras ($*$-wildness), it was suggested in [11] to choose, for a standard difficult problem in the theory of $*$-representations, the problem of describing pairs of self-adjoint (or unitary) operators up to unitary equivalence (free $*$-algebras $\mathfrak{S}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*\rangle$ (or $\mathfrak{U}_2 = \mathbb{C}\langle u, v, u^*, v^* \mid uu^* = u^*u = e, vv^* = v^*v = e\rangle$) generated by pair of self-adjoint (or unitary) generators), and there were indications to consider problems, which contains the standard $*$-wild problem, $*$-wild – one can prove that these problems contain as a subproblem the problem of describing $*$-representations of any affine $*$-algebra. q We give exact deffinition.

Definition 1 *[16] Let $\mathfrak{A}$ be a $*$-algebra. A pair $(\widetilde{\mathfrak{A}} : \phi : \mathfrak{A} \to \widetilde{\mathfrak{A}})$, where $\widetilde{\mathfrak{A}}$ is a $*$-algebra and ϕ is a $*$-homomorphism, is called an enveloping $*$-algebra of the algebra $\mathfrak{A}$ if, for any $*$-representation $\pi : \mathfrak{A} \to L(H)$ of the algebra $\mathfrak{A}$, there exists a unique $*$-representation $\tilde{\pi} : \widetilde{\mathfrak{A}} \to L(H)$ such that $\pi = \tilde{\pi} \circ \phi$*

Now, we give some examples of enveloping $*$-algebras:

1) $\widetilde{\mathfrak{A}} = \mathfrak{A}$, ϕ is the identity mapping;

2) Let Σ be any set of elements of any algebra $\mathfrak{A}$, the images of which are invertible operators for any representation $\pi : \mathfrak{A} \to L(H)$. Let $\widetilde{\mathfrak{A}} = \mathfrak{A}[\Sigma^{-1}]$ be the quotient algebra (see [8]) of the algebra $\mathfrak{A}$ wich respect to the set Σ, and let ϕ be the natural imbedding of $\mathfrak{A}$ into $\mathfrak{A}[\Sigma^{-1}]$;

3)Let $\mathfrak{A}$ be a $*$-bounded $*$-algebra, $\widetilde{\mathfrak{A}}$ be its enveloping C^*-algebra, ϕ – its canonical $*$-homomorphism of $\mathfrak{A}$ into $\widetilde{\mathfrak{A}}$ defined by a faithful representation (see, for example, [10]).

Let $M_n(\widetilde{\mathfrak{A}})$ be the matrix algebra over $\widetilde{\mathfrak{A}}$ with the naturally given $*$-structure. Any representation $\tilde{\pi} : \widetilde{\mathfrak{A}} \to L(H)$ induced the representation $\tilde{\pi}_n : M_n(\widetilde{\mathfrak{A}}) \to L(H \oplus H \cdots \oplus H)$. By $Rep(\mathfrak{A})$ we denote the category, objects of which are representations of the $*$-algebra $\mathfrak{A}$ and morphisms – intertwining operators. If $\psi : \mathfrak{B} \to M_n(\widetilde{\mathfrak{A}})$ is a $*$-homomorphism, then there is a natural way to construct the functor $F_\psi : Rep(\mathfrak{A}) \to Rep(\mathfrak{B})$. By definition, $F_\psi = \tilde{\pi}_n \circ \psi$ and, if $\alpha : \pi \to \pi_1$ is a morphism of representations, then $F_\psi(\alpha) = diag(\alpha, \alpha, \ldots, \alpha)$.

Definition 2 *[16] A $*$-algebra $\mathfrak{B}$ majoriza a $*$-algebra $\mathfrak{A}$ ($\mathfrak{B} \succ \mathfrak{A}$) if there exist $n = 1, 2, \ldots$, an enveloping algebra $\widetilde{\mathfrak{A}}$ of the algebra $\mathfrak{A}$, and a $*$-homomorphism $\psi : \mathfrak{B} \to M_n(\widetilde{\mathfrak{A}})$ such that the functor $F_\psi : Rep(\mathfrak{A}) \to Rep(\mathfrak{B})$ is full and faithful.*

In this case we will say that the problem of unitary classification of representations of the $*$-algebra $\mathfrak{B}$ contains, as a subproblem, the problem of a unitary classification of representations of the algebra $\mathfrak{A}$. It follows from definition that two representation π_1 and π_2 of the algebra $\mathfrak{A}$ are unitary equivalent (irreducible) if and only if the representations $F_\psi(\pi_1)$ and $F_\psi(\pi_2)$ are unitary equivalent (irreducible).

The relation "$\succ$" induce by natural way a quasiorder for a $*$-algebras . This quasiorder will be used in the siqual, its also will be called a majoration and denote by "$\succ$".

As a model of complexity for problems of unitary classification of representations of the $*$-algebra one can choose, for example, the problem of unitary classification of representation of the $*$-algebra $\mathfrak{U}_2 = \mathbb{C}\langle u, v, u^*, v^* \mid uu^* = u^*u = e, \quad vv^* = v^*v = e\rangle$ or which is the same thing, the problem of unitary classification of the representations of its enveloping C^*-algebra $C^*(\mathcal{F}_2)$, where $\mathcal{F}_2$ is a free group with two generators (see [11]) ($\mathfrak{A} \succ \mathfrak{U}_2$ iff $\mathfrak{A} \succ C^*(\mathcal{F}_2)$).

Definition 3 *[16] A $*$-algebra $\mathfrak{A}$ as called $*$-wild if $\mathfrak{A} \succ C^*(\mathcal{F}_2)$.*

Late on, for proof, that a $*$-algebra $\mathfrak{A}$ is $*$-wild we will show that $\mathfrak{A} \succ \mathfrak{U}_2$.

One can prove that if a $*$-algebra $\mathfrak{A}$ is $*$-wild then that is the $*$-algebra of non-type I (see [13, 14]). The converse statment is not true. Thus, for example, the algebra of Cuntz O_n ($n \geq 2$) is the algebra of non-type I which is not $* - wild$ ([13, 14].

3 On unitary classification of non-selfadjoint operators

3.1 On weakly centered operators

We shall now study weakly centered operators and partial isometry operators, and weakly centered operators which are partial isometry operators. We will consider the operators X and X^* which satisfy the relations $P_j(X, X^*) = 0$ $(j = 1, \ldots, m)$, as representation of the $*$-algebra $\mathfrak{A} = \langle x, x^* \mid P_j(x, x^*) = 0\rangle$. Thus, we will study the unitary classification of representations of the $*$-algebras $\mathfrak{V} = \mathbb{C}\langle x, x^* \mid [xx^*, x^*x] = 0\rangle$, $\mathfrak{N} = \mathbb{C}\langle y, y^* \mid (y^*y)^2 = y^*y\rangle$ and $\mathfrak{M} = \mathbb{C}\langle x, x^* \mid [xx^*, x^*x] = 0, (x^*x)^2 = x^*x\rangle$.

Theorem 1 *The $*$-algebra $\mathfrak{V}$ is $*$-wild.*

Proof. We have to show that $\mathfrak{V} \succ \mathfrak{U}_2$, where $\mathfrak{U}_2 = \mathbb{C}\langle u, v, u^*, v^* \mid uu^* = u^*u = e, \quad vv^* = v^*v = e\rangle$.

We give the $*$-homomorphism $\psi : \mathfrak{V} \to M_3(\mathfrak{U}_2)$ of the $*$-algebra $\mathfrak{V}$ into the $*$-algebra $\mathfrak{U}_2$ as

$$\psi(x) = \begin{bmatrix} 0 & 0 & 2e \\ (1/2)e & (\sqrt{3}/2)v & 0 \\ (\sqrt{3}/2)u & -(1/2)uv & 0 \end{bmatrix}.$$

To show that this is a $*$-homomorphism, we calculate that

$$\psi(x)\psi(x^*) = \begin{bmatrix} 2e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}, \psi(x^*)\psi(x) = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 2e \end{bmatrix}.$$

Therefore $[\psi(x)\psi(x^*), \psi(x^*)\psi(x)] = 0$. The $*$-homomorphism ψ induces the functor $F_\psi : Rep(\mathfrak{U}_2) \to Rep(\mathfrak{A})$ as follows

- if $\rho \in Ob(Rep(\mathfrak{U}_2))$: $\rho(u) = U$ and $\rho(v) = V$, then $F_\psi(\rho) = \rho \circ \psi = \pi$, where $\pi(x) = X$ and $\pi(x^*) = X^*$;
- if $\alpha : \rho \to \hat{\rho}$ (that is $\alpha U = \hat{U}\alpha, \quad \alpha V = \hat{V}\alpha$) , then $F_\pi(\alpha) = diag(\alpha, \alpha, \alpha)$ and $F_\psi(\alpha) : \pi \to \hat{\pi}$ (that is $F_\psi(\alpha)X = \hat{X}F_\psi(\alpha)$, $F_\psi(\alpha)X^* = \hat{X}^*F_\psi(\alpha)$).

It is evident that the functor F_ψ is faithful. We will show that F_ψ is full.

It follows from $F_\psi(\alpha)X^*X = \hat{X}^*\hat{X}F_\psi(\alpha)$ that

$$F_\psi(\alpha) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}.$$

¿From the relations $F_\psi(\alpha)X = \hat{X}F_\psi(\alpha)$, $F_\psi(\alpha)X^* = \hat{X}^*F_\psi(\alpha)$ we have that $\alpha_{12} = \alpha_{21} = 0$, $\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha$ and $\alpha U = \hat{U}\alpha$, $\alpha V = \hat{V}\alpha$. Hence, we can conclude that the functor F_ψ is full. Therefore, the algebra $\mathfrak{V}$ is $*$-wild. Q.e.d.

Theorem 2 *The $*$-algebra $\mathfrak{N}$ is $*$-wild.*

We will show that $\mathfrak{N} \succ \mathfrak{U}_2$. The $*$-homomorphism $\psi : \mathfrak{N} \to M_3(\mathfrak{U}_2)$ is

$$\psi(y) = \begin{bmatrix} (\sqrt{3}/4)u & (\sqrt{3}/2)e & 0 \\ (3/4)v & -(1/2)vu^* & 0 \\ (1/2)e & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $\psi(y^*)\psi(y) = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $(\psi(y^*)\psi(y))^2 = \psi(y^*)\psi(y)$, hence the $*$-homomorphism ψ has been defined correctly.

The proof that the induced functor $F_\psi : Rep(\mathfrak{P}) \to Rep(\mathfrak{B})$ $(\rho \to \rho \circ \psi; \alpha \to diag(\alpha, \alpha, \alpha)$ is full and faithful is similar to the proof in theorem 1. Q.e.d.

Theorem 3 *The $*$-algebra $\mathfrak{M}$ is $*$-wild.*

Proof. We will again shown that $\mathfrak{M} \succ \mathfrak{U}_2$. We give the $*$-homomorphism $\psi : \mathfrak{M} \rightarrow M_4(\mathfrak{U}_2)$ as follows:

$$\psi(z) = \begin{bmatrix} (\sqrt{3}/4)u & (\sqrt{3}/2)e & 0 & 0 \\ (3/4)v & -(1/2)vu^* & 0 & 0 \\ (1/2)e & 0 & 0 & 0 \\ 0 & 0 & e & 0 \end{bmatrix}.$$

As in theorem 1 we can show that the induced functor $F_\psi : Rep(\mathfrak{U}_2) \rightarrow Rep(\mathfrak{N})$ is full and faithful. Q.e.d.

3.2 On Centered operators

Now we will consider a subclass of weakly centered operators which are not $*$-wild. That is the centered operators. Let the corresponding $*$-algebra be $\mathfrak{C} = \mathbb{C}\langle x, x^* \mid \forall i, j[x^i, (x^*)^i, x^j(x^*)^j] = [x^i(x^*)^i, (x^*)^j, x^j] = [(x^*)^i, x^i, (x^*)^j x^j] = 0\rangle$.

Proposition 1 *The $*$-algebra $\mathfrak{C}$ is not $*$-wild.*

Proof. Let π be a representation of $\mathfrak{C}$, where $\pi(x) = X$ and $\pi(x^*) = X^*$ on a Hilbert space H such that $\langle X, X^*\rangle''$ is factor of type II. Here $\langle X, X^*\rangle''$ is a von Neumann algebra generating by operators of representations. Let $X = UC$ be a polar decomposition. We can suppose, that U is unitary [20]. In [19] it is proved that X is centered and quasi-invertible (U is unitary) if and only if the infinite sequence $\{U^n C(U^*)^n\}$ consists of mutually commuting operators. We consider the commutative algebra $\mathfrak{A} = \langle U^k C(U^*)^k, k \in \mathbb{Z}\rangle''$. The operator U gives a free, ergodic action of the group $\mathbb{Z}$ on $\mathfrak{A}$. Therefore $\langle U, U^* C(U^*)^k, k \in \mathbb{Z}\rangle''$ is a cross product of commutative von Neumann algebra $\mathfrak{A}$ and group $\mathbb{Z}$. It follows from the general theory of von Neumann algebras, that this cross product is a hyperfinite factor (see [6]). The algebra $\langle X, X^*\rangle'' = \langle U, U^* C(U^*)^k, k \in \mathbb{Z}\rangle''$. Thus every factor-representation of $\mathfrak{C}$ of type II is hyperfinite, but that is not true for $*$-wild algebras [14]. Therefore, the $*$-algebra $\mathfrak{C}$ is not $*$-wild. Q.e.d.

Remark. Let operators X and X^* acting in a Hilbert space H satisfy the conditions: $\forall i, j = 1, \ldots n$ $(n < \infty)$

$$[X^i(X^*)^i, X^j(X^*)^j] = [X^i, (X^*)^i, (X^*)^j, X^j] = [(X^*)^i, X^i, (X^*)^j X^j] = 0$$

(this class is situated between the centered and weakly centered operators). Then the problem to desribe such class of operators up to unitary equivalence is $*$-wild [1]. The same holds even the X is partial isometry [1].

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