MSC 11M36

Evaluations of higher depth determinants of Laplacians [1]

© **Y. Yamasaki**
Ehime University, Matsuyama, Japan

In this article, we evaluate "higher depth determinants" of Laplacians on the compact Riemann surfaces with negative constant curvature. This is a summary of the results in the forthcoming paper [5]

Keywords: Hurwitz's zeta function, Selberg's zeta function, multiple gamma function, polylogarithm function, determinants of Laplacians

§ 1. Introduction

Let T be an operator on some space. We assume that T has only discrete spectrum $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots \to +\infty$ and the multiplicity of each eigenvalue λ_j is finite. Define the spectral zeta function $\zeta_T(\omega, z)$ attached to T of Hurwitz's type by

$$\zeta_T(w,z) := \sum_{j=0}^{\infty}(\lambda_j + z)^{-w}.$$

We assume that the series converges absolutely in some right half w-plane (uniformly for z on any compact set) and can be continued meromorphically to a region containing $w = 1 - r$ for $r \in \mathbb{N}$. Moreover, we assume that $\zeta_T(w, z)$ is holomorphic at $w = 1 - r$. In this case, we define a *higher depth determinant* of T of depth r by

$$\mathrm{Det}_r(T+z) := \exp\left(-\frac{\partial}{\partial w}\,\zeta_T(w,z)\Big|_{w=1-r}\right).$$

This can be considered as a determinant analogue of the Milnor gamma function

$$\mathbf{\Gamma}_r(z) := \exp(\frac{\partial}{\partial w}\zeta(w,z)|_{w=1-r})$$

studied in [7] (see also [4]). Here $\zeta(w,z) := \sum_{n=0}^{\infty}(n+z)^{-w}$ is the Hurwitz zeta function. Note that $\det(T + z) := \mathrm{Det}_1(T + z)$ (with $z = 0$) gives the usual "normalized determinant" of T (see, e.g., [9]).

[1] This work is partially supported by Grant-in-Aid for JSPS Fellows No.19002485

The aim of the present paper is to evaluate the higher depth determinants when T is the Laplacian Δ_Γ on the compact Riemann surface $\mathcal{R} = \Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the complex upper half plane with the standard Poincaré metric and Γ is a discrete, co-compact torsion-free subgroup of $\mathrm{SL}_2(\mathbb{R})$. We show the following theorem, which gives a generalization of the result for $r = 1$ in [11] (see also [8]).

Theorem 1.1 [**5**] *The higher depth determinants* $\mathrm{Det}_r(\Delta_\Gamma - s(1-s))$ *can be explicitly expressed as a product of the multiple gamma functions and "Milnor-Selberg zeta functions".*

This is a summary article; readers who are interested in this topic can find the detailed proof of the above result in the forthcoming paper [5]. See also [13] for explicit calculations of the higher depth determinants of the Laplacian on spheres in higher dimensions.

§ 2. A product expression of $\mathrm{Det}_r(\Delta_\Gamma - s(1-s))$

We first recall the Selberg trace formula for the Riemann surface $\mathcal{R} = \Gamma\backslash\mathbb{H}$ of genus $g \geqslant 2$. Let f be a function whose Fourier transform

$$\widehat{f}(r) := \int_{-\infty}^{\infty} f(x) e^{-irx} dx$$

satisfies the conditions $\widehat{f}(-r) = \widehat{f}(r)$, $\widehat{f}$ is holomorphic in the band $|\mathrm{Im}\, r| < \delta + 1/2$ and $\widehat{f}(r) = O(|r|^{-2-\delta})$ as $|r| \to \infty$ for some $\delta > 0$. Then, the formula reads

$$\begin{aligned} \sum_{j=0}^{\infty} m_j \widehat{f}(r_j) &= \sum_{\gamma \in \mathrm{Hyp}(\Gamma)} \frac{\log N(\delta_\gamma)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} f(\log N(\gamma)) \\ &+ (g-1) \int_{-\infty}^{\infty} \widehat{f}(r)\, r \tanh(\pi r)\, dr. \end{aligned} \tag{2.1}$$

Here m_j is the multiplicity of λ_j ($j \geqslant 0$), r_j is the number determined by $\lambda_j = \frac{1}{4} + r_j^2$ ($r_j \geqslant 0$ if $r_j \in \mathbb{R}$ and $-ir_j > 0$ otherwise), $\mathrm{Hyp}(\Gamma)$ (resp. $\mathrm{Prim}\,(\Gamma)$) is the set of all hyperbolic (resp. primitive) conjugacy classes in Γ, $N(\gamma)$ is the square of the larger eigenvalue of $\gamma \in \mathrm{Hyp}(\Gamma)$ and, for $\gamma \in \mathrm{Hyp}(\Gamma)$, $\delta_\gamma \in \mathrm{Prim}\,(\Gamma)$ is the unique element satisfying $\gamma = \delta_\gamma^k$ for some $k \geqslant 1$.

Suppose $\mathrm{Re}\, s > 1/2$ and $\mathrm{Re}\, w > r$. Write $t = s - 1/2$. Let

$$f(x) := \frac{1}{\sqrt{\pi}\,\Gamma(w+1-r)} \left(\frac{x}{2t}\right)^{w-r+1/2} K_{w-r+1/2}(tx),$$

where $\Gamma(x)$ is the classical gamma function and $K_\nu(x)$ is the K-Bessel function. Then, taking this f as a test function of the trace formula (2.1) and noticing that

$\widehat{f}(r) = (r^2+t^2)^{-w+r-1}$, we have

$$\begin{aligned}\zeta_{\Delta_\Gamma}\bigl(w+1-r,\ -s(1-s)\bigr) &= I_r\bigl(w,t\bigr) + \\ + \quad (g-1)\sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\, &t^{2(r-1-\ell)}J^{(2\ell+1)}(w,t),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} I_r(w,a) &:= \sum_{\gamma\in \mathrm{Hyp}(\Gamma)} \frac{\log N(\delta_\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}\, A_r(w,a;\gamma), \\ A_r(w,a;\gamma) &:= \frac{1}{\sqrt{\pi}\,\Gamma(w-r+1)}\left(\frac{\log N(\gamma)}{2a}\right)^{w-r+1/2} K_{w-r+1/2}\bigl(a\log N(\gamma)\bigr)\end{aligned}$$

and

$$J^{(m)}(w,a) := \int_{-\infty}^{\infty} (x^2+a^2)^{-w}\, x^m\, \tanh(\pi x)\, dx.$$

One can show that both $I_r(w,a)$ and $J^{(m)}(w,a)$ are continued meromorphically to $\mathbb{C}$ as a function of w and are in particular holomorphic at $w=0$. Hence, taking the derivatives at $w=0$ of the both hands side of (2.2), we have

$$\mathrm{Det}_r\bigl(\Delta_\Gamma - s(1-s)\bigr) = \phi_r(s)^{g-1} Z_{\Gamma,r}(s), \tag{2.3}$$

where

$$\phi_r(s) := \prod_{\ell=0}^{r-1} \exp\Bigl(-\frac{\partial}{\partial w}J^{(2\ell+1)}(w,t)\Big|_{w=0}\Bigr)^{\binom{r-1}{\ell}t^{2(r-1-\ell)}}, \tag{2.4}$$

$$Z_{\Gamma,r}(s) := \exp\Bigl(-\frac{\partial}{\partial w}I_r(w,t)\Big|_{w=0}\Bigr). \tag{2.5}$$

In the subsequence sections, we calculate the "gamma factor" $\phi_r(s)$ (Theorem 3.1) and the "zeta factor" $Z_{\Gamma,r}(s)$ (Theorem 4.1), respectively. As a consequence, our main result (Theorem 1.1) follows immediately from the equation (2.3).

§ 3. Gamma factor

To evaluate $\phi_r(s)$, we here recall the Barnes multiple gamma functions. Let

$$\zeta_n(w,z) := \sum_{m_1,\dots,m_n\geqslant 0} \frac{1}{(m_1+\ldots+m_n+z)^w}, \qquad \mathrm{Re}\, w > n,$$

be the Barnes multiple zeta function [2]. It is known that $\zeta_n(w,z)$ can be continued meromorphically to $\mathbb{C}$ with possible simple poles at $w=1,2,\dots,n$. Then, the Barnes multiple gamma function $\Gamma_n(z)$ is defined by

$$\Gamma_n(z) := \exp\left(\frac{\partial}{\partial w}\,\zeta_n(w,z)\Big|_{w=0}\right).$$

Note that $\Gamma_1(z) = \Gamma(z)/\sqrt{2\pi}$ from the Lerch formula [6]

$$\frac{\partial}{\partial w}\,\zeta(w,z)\Big|_{w=0} = \log\,\frac{\Gamma(z)}{\sqrt{2\pi}}\,.$$

One can evaluate the integral $J^{(m)}(w,a)$ by using the residue theorem and its derivative at $w = 0$ in terms of (the logarithm of) the Barnes multiple gamma function. Consequently, together with the formula (2.4), we have the following expression of $\phi_r(s)$;

Theorem 3.1 *Write $t = s - 1/2$. Then we have*

$$\phi_r(s) = \exp\left\{-\frac{(2r)!!}{r^2\,(2r-1)!!}\,t^{2r}\right\}\cdot\prod_{j=1}^{2r}\Gamma_j(s)^{\alpha_{r,j}(t)},$$

where $\alpha_{r,j}(t)$ is the even polynomial given by

$$\alpha_{r,j}(t) := 4\sum_{\ell=[\frac{j-1}{2}]}^{r-1}\binom{r-1}{\ell}(-1)^{\ell}\,c_{2\ell+2,j}\left(\frac{1}{2}\right)\,t^{2(r-1-\ell)}\,.$$

Here $[x]$ denotes the largest integer not exceeding x and $c_{r,j}(z)$ is the polynomial defined by

$$(T+z)^{r-1} = \sum_{j=1}^{r}c_{r,j}(z)\binom{T+j-1}{j-1}.$$

Example 3.2 Write $t = s - 1/2$. Then it holds that

$$\begin{aligned}
\phi_1(s) &= e^{-2t^2}\Gamma_1(s)^{-2}\Gamma_2(s)^4,\\
\phi_2(s) &= e^{-\frac{2}{3}t^4}\Gamma_1(s)^{-2t^2+\frac{1}{2}}\Gamma_2(s)^{4t^2-13}\Gamma_3(s)^{36}\Gamma_4(s)^{-24},\\
\phi_3(s) &= e^{-\frac{16}{45}t^6}\Gamma_1(s)^{-\frac{1}{8}+t^2-2t^4}\Gamma_2(s)^{\frac{121}{4}-26t^2+4t^4}\Gamma_3(s)^{-330+72t^2}\\
&\quad\times\Gamma_4(s)^{1020-48t^2}\Gamma_5(s)^{-1200}\Gamma_6(s)^{480}.
\end{aligned}$$

Remark 3.3 The function $\phi_r(s)$ can be also expressed in terms of the Vignéras multiple gamma functions $G_n(z)$ [12], see also [1] for the case $n = 2$, which are characterized by a generalization of the Bohr-Mollerup theorem. We remark that $\Gamma_n(z)$ and $G_n(z)$ are essentially equal (see, more precisely, [10]).

§ 4. Zeta factor

We next evaluate $Z_{\Gamma,r}(s)$. To do that, we introduce a *Milnor-Selberg zeta function* $Z_{\Gamma}^{(m)}(s)$ of depth m by the following Euler product:

$$Z_{\Gamma}^{(m)}(s) := \prod_{P\in\mathrm{Prim}\,(\Gamma)}\prod_{n=0}^{\infty}\mathrm{H}_m\left(N(P)^{-s-n}\right)^{(\log N(P))^{-m+1}},\qquad \mathrm{Re}\,s>1. \qquad (4.1)$$

Here $\mathrm{H}_m(z) := \exp(-\mathrm{Li}_m(z))$ with

$$\mathrm{Li}_m(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^m}$$

being the polylogarithm function. Notice that, since $\mathrm{Li}_1(z) = -\log(1-z)$, this gives the Selberg zeta function

$$Z_\Gamma(s) := \prod_{P \in \mathrm{Prim}\,(\Gamma)} \prod_{n=0}^{\infty} \big(1 - N(P)^{-s-n}\big), \qquad \mathrm{Re}\, s > 1.$$

We remark that, since

$$\frac{d}{dz}\mathrm{Li}_m(z) = \frac{1}{z}\mathrm{Li}_{m-1}(z),$$

the Milnor-Selberg zeta functions $Z_\Gamma^{(m)}(s)$ satisfy the following differential ladder relation;

$$\frac{d^{m-1}}{ds^{m-1}} \log Z_\Gamma^{(m)}(s) = -\frac{d^{m-2}}{ds^{m-2}} \log Z_\Gamma^{(m-1)}(s) = \ldots = (-1)^{m-1} \log Z_\Gamma(s).$$

This shows that the Milnor-Selberg zeta function of depth m is essentially given by the $(m-1)$-th iterated integrals of the logarithm of the Selberg zeta function $Z_\Gamma(s)$. Therefore, since $Z_\Gamma(s)$ has zeros at $s = -k$ for $k = -1, 0, 1, 2, \ldots$ and $(1/2) \pm ir_j$ for $j = 1, 2, \ldots$, it is in general a multi-valued function.

Since the K-Bessel function $K_\nu(x)$ is analytic with respect to the variable ν, one can see that the functions $A_r(w, a; \gamma)$ and hence $I_r(w, a)$ are holomorphic at $w = 0$. Moreover, using the asymptotic formulas

$$\begin{aligned} K_{w-r+1/2}(a \log N(\gamma)) &= K_{-r+1/2}\big(a \log N(\gamma)\big) + O(w), \\ \left(\frac{\log N(\gamma)}{2a}\right)^{w-r+1/2} &= \left(\frac{\log N(\gamma)}{2a}\right)^{-r+1/2} + O(w), \\ \frac{1}{\Gamma(w-r+1)} &= (-1)^{r-1}\,(r-1)!\, w + O(w^2) \end{aligned}$$

as $w \to 0$ together with the well-known formula, see, e. g., [3],

$$K_{-r+1/2}(y) = K_{r-1/2}(y) = \left(\frac{\pi}{2y}\right)^{1/2} e^{-y} \sum_{m=0}^{r-1} (2y)^{-m} \frac{(r+m-1)!}{m!(r-m-1)!}\,,$$

we have

$$\begin{aligned} \frac{\partial}{\partial w} I_r(w, a)\Big|_{w=0} \;&=\; (-1)^{r-1} \sum_{m=0}^{r-1} \binom{r-1}{m} (r+m-1)! \,\times \\ &\times\; (2a)^{r-1-m}\, z_\Gamma\left(a + \frac{1}{2}\,,\, r+m\right), \end{aligned} \tag{4.2}$$

where

$$z_\Gamma(s,m) := \sum_{\gamma\in \mathrm{Hyp}(\Gamma)} \frac{\log N(\delta_\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} \cdot \frac{N(\gamma)^{-s+1/2}}{(\log N(\gamma))^m}.$$

By a straightforward calculation, one can show that

$$\log Z_\Gamma^{(m)}(s) = -z_\Gamma(s,m).$$

Therefore, together with the formulas (2.5) and (4.2), we obtain the following

Theorem 4.1 *Write $t = s - 1/2$. Then we have*

$$Z_{\Gamma,r}(s) = \left(\prod_{m=0}^{r-1} Z_\Gamma^{(r+m)}(s)^{\binom{r-1}{m}(r+m-1)!(2t)^{r-1-m}} \right)^{(-1)^{r-1}}, \quad \mathrm{Re}\, s > 1.$$

Example 4.2 Write $t = s - \frac{1}{2}$. Then it holds that

$$\begin{aligned} Z_{\Gamma,1}(s) &= Z_\Gamma^{(1)}(s) = Z_\Gamma(s), \\ Z_{\Gamma,2}(s) &= Z_\Gamma^{(2)}(s)^{-2t} Z_\Gamma^{(3)}(s)^{-2}, \\ Z_{\Gamma,3}(s) &= Z_\Gamma^{(3)}(s)^{8t^2} Z_\Gamma^{(4)}(s)^{24t} Z_\Gamma^{(5)}(s)^{24}. \end{aligned}$$

Remark 4.3 One can see that the function $Z_{\Gamma,1}(s)$ is an entire function because the Selberg zeta function $Z_\Gamma(s)$ is (notice that $Z_{\Gamma,1}(s) = Z_\Gamma^{(1)}(s) = Z_\Gamma(s)$). However, for $r \geqslant 2$, it has not been clarified whether $Z_{\Gamma,r}(s)$, which is written as a product of $Z_\Gamma^{(m)}(s)$, can be (analytically) continued to a single-valued function (recall that the Milnor-Selberg zeta function $Z_\Gamma^{(m)}(s)$ is in general a multi-valued function).

References

1. E. W. Barnes. The theory of the G-function, Quart. J. Math., 1899, vol. 31, 264–314.

2. E. W. Barnes. On the theory of the multiple gamma functions, Trans. Cambridge Philos. Soc., 1904, vol. 19, 374–425.

3. A. Erdélyi, W. Magnus, F. Oberthettinger and F. G. Tricomi. Higher Transcendental Functions, McGraw-Hill, New York, 1953.

4. N. Kurokawa, H. Ochiai and M. Wakayama. Milnor's multiple gamma functions, J. Ramanujan Math. Soc., 2002, vol. 21, 153–167.

5. N. Kurokawa, M. Wakayama and Y. Yamasaki. Higher depth determinants of Laplacians and Milnor-Selberg zeta functions, preprint, 2008.

6. M. Lerch. Dalši studie v oboru Malmsténovských řad, Rozpravy České Akad., 1894, vol. 3, No. 28, 1–61.

7. J. Milnor. On polylogarithms, Hurwitz zeta functions, and the Kubert identities, Enseignement Mathématique, 1983, vol. 29, 281–322.

8. P. Sarnak. Determinants of Laplacians, Commun. Math. Phys., 1987, vol. 110, 113–120.

9. C. Soulé, D. Abramovich, J.-F. Burnol and J. Kramer. Lectures on Arakelov geometry, Cambridge Studies in Advanced Mathematics, 33. Cambridge University Press, Cambridge, 1992.

10. H. M. Srivastava and J. Choi. Series associated with the zeta and related functions, Kluwer Academic Publishers, Dordrecht, 2001.

11. A. Voros. Spectral functions, special functions and the Selberg zeta functions, Commun. Math. Phys., 1987, vol. 110, 439–465.

12. M. F. Vignéras. L'équation fonctionelle de la fonction zeta de Selberg de groupe modulaire $PSL(2, \mathbb{Z})$, Astérisque, 1979, vol. 61, 235–249.

13. Y. Yamasaki. Higher depth determinants of the Laplacian on n-sphere. Preprint, 2008.