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On normalization of representations [1]

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It is well known that principal series representations T_s of the group $\mathrm{SL}(2,\mathbb{R})$ satisfy the condition $T_s \simeq T_{-s}$ for $s \neq \pm 1, \pm 2, \ldots$. In 1960 Kunze and Stein constructed a family of representations $V(s)$, holomorphic in the strip $|\mathrm{Re}\, s| < 1$ and satisfying the condition: $V(s) = V(-s)$, $V(s) \simeq T_s$, $|\mathrm{Re}\, s| < 1$; this family was called a *normalized family*. In the paper we discuss different modifications of this subject and apply normalized families to construct representations of rather general kind

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§ 1. Example

We begin with the following example in order to explain a notion "normalization". Let a group G act on a connect complex manifold X by holomorphic transformations $x \mapsto xg$. Let $x \mapsto f(x)$, $x \in X$, be a holomorphic function with values in $n \times n$ matrices. Suppose that on some open subset $X_0 \subset X$ the matrix $f(x)$ has simple eigenvalues and $f(xg) \simeq f(x)$ for any $x \in X_0$ (here $\simeq$ stands for matrix similarity). We are going to construct another holomorphic function $x \mapsto f_1(x)$, $x \in X$, with values in $n \times n$ matrices, such that $f(x) \simeq f_1(x)$ on some open dence subset $X_1 \subset X$ and $f_1(xg) = f_1(x)$, $x \in X$, $g \in G$. To do it, we consider the characteristic polynomial of the matrix $f(x)$:

$$\det(\lambda E - f(x)) = \lambda^n + d_{n-1}(x)\,\lambda^{n-1} + \ldots + d_1(x)\,\lambda + d_0(x)$$

and put

$$f_1(x) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -d_0(x) \\ 1 & 0 & \ldots & 0 & -d_1(x) \\ \vdots & \vdots & \vdots\vdots & \vdots & \\ 0 & 0 & \ldots & 1 & -d_{n-1}(x) \end{pmatrix}.$$

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This function satisfies our requirement. It is called a *normalization* of the initial function.

§ 2. Normalization of representations

The first example of normalization of representations was constructed by Kunze and Stein in [1] for the group $\mathrm{SL}(2,\mathbb{R})$.

It is convenient to replace this group by the isomorphic group $G = \mathrm{SU}(1,1)$. Denote by Γ the circle $\{u \in \mathbb{C}, |u| = 1\}$. The representations T_s, $s \in \mathbb{C}$, of the group G act on the space $L^2(\Gamma)$ by

$$(T_s(g)f)(u) = |bu + \overline{a}|^{s-1} f\left(\frac{au + \overline{b}}{bu + \overline{a}}\right), \quad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}. \tag{1}$$

There is a symmetry relation: $T_{-s} \simeq T_s$ if $s \neq \pm 1, \pm 2, \ldots$.

Kunze and Stein have constructed a holomorphic family of representations $V(s)$ for the strip $\{s \in \mathbb{C} : |\mathrm{Re}\, s| < 1\}$, such that $V(s) \simeq T_s$ and $V(s) = V(-s)$ for all s in this strip. Later the author constructed a normalization $V(s,a)$, where $a > 1$ is an arbitrary number, for the strip $\{s \in \mathbb{C} : |\mathrm{Re}\, s| < a\}$, see [2]. This strip contains "singular points" $\pm 1, \pm 2, \ldots$, so that $V(s,a) \simeq T_s$, $V(s,a) = V(-s,a)$ for all s in this strip except $s = \pm 1, \pm 2, \ldots$.

The similar prolem can be also posed for other groups. In [3] the group $\mathrm{Mot}(\mathbb{R}^n)$ (motions of the space $\mathbb{R}^n$) was considered. In this case the normalized family is holomorphic on the whole complex plane. Recently the authors of [3] have considered the group of hyperbolic motions.

§ 3. Applications: construction of "arbitrary" spherical representations

Consider the group $G = \mathrm{SU}(1,1)$ and its representation T by bounded operators in a Hilbert space H. Assume that this representation is spherical with respect to the rotation subgroup K (diagonal matrices). It means that the representation T is unitary on K and the subspace $H_0 = \{h : T(k)h = h, k \in K\}$ generates H. The question is: how to "describe all" such representations?

We do not try to set the problem in strict terms. Instead we consider the following (rather natural) approach to this problem.

Consider the Laplacian Δ on G, its image $T(\Delta)$ under T and the restriction A of $T(\Delta)$ to H_0. Thus the spherical representation T results in a pair (H_0, A). It is easy to prove that the spectrum of A lies in the domain $D_a = \{z = x+iy, x < -(y/2a)^2+a^2\}$ with some a and for points $z \notin D_a$ we have $\|(A - zI)^{-1}\| < C\,|\mathrm{Re}\,(\sqrt{z})|$.

We now ask: is it possible to reconstruct the representation T by using the pair (H_0, A)? This is an "inverse representation problem" for our group G.

Before discussing this problem, let us begin with a very simple construction of spherical representations. Consider the Hilbert space $L^2(\Gamma, H_0)$ of vector-functions, an operator S in H_0 and define a representation T_S by the formula (1) where the number s is replaced by this operator S. The operator S is assumed to generate an one-parameter subgroup $t \mapsto \exp(tS)$. One would think that we obtain a sufficently big family of representations. However such an impression is false. The reason is the following. It is easy to show that the operator A which corresponds to this representation (in the sense just explained) has the form $A = (S/2)^2 - (1/4)I$, so that the operator $4A + I$ admits a square root. It is known however that there exist operators even bounded which have no square root. So we see that the operators T_S do not solve our "inverse problem".

The situation becomes more tractable if we use the normalized family $V(a, s)$ with sufficiently big parameter a. Since the function $s \mapsto V(s, a)$ is even (and hence is a holomorphic function on s^2), we can substitute $\sqrt{4A + I}$ instead of s (although the operator may $4A + I$ have no square roots). This leads to a solution of our "inverse problem". (Of course this substitution is possible under some restrictions; in particular the spectrum of A must lie in some domain D_a. These restrictions are discussed in [2].)

The "inverse"problem for the group $\mathrm{Mot}(\mathbb{R}^n)$ was investigated in [3].

The authors of [3] are going to discuss the further development of this subject in a forthcoming paper.

References

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