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Properties of the algebra Psd related to integrable hierarchies

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Свойства алгебры псевдодифференциальных операторов, связанные с интегрируемыми иерархиями

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Abstract. In this paper we discuss and prove various properties of the algebra of pseudo differential operators related to integrable hierarchies in this algebra, in particular the KP hierarchy and its strict version. Some explain the form of the equations involved or give insight in why certain equations in these systems are combined, others lead to additional properties of these systems like a characterization of the eigenfunctions of the linearizations of the mentioned hierarchies, the description of elementary Darboux transformations of both hierarchies and the search for expressions in Fredholm determinants for the constructed eigenfunctions and their duals.

Keywords: pseudo differential operators; the adjoint; constant term; n-KdV hierarchy; KP hierarchy; strict KP hierarchy; Lax equations

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Аннотация. В работе рассматриваются различные свойства алгебры псевдодифференциальных операторов, связанные с интегрируемыми иерархиями, возникающими в этой алгебре, в частности, иерархией Кадомцева–Петвиашвили (КП) и ее строгой версией. Одни свойства проясняют вид уравнений в иерархиях и дают понимание того, почему уравнения определенного вида скомбинированы в этих системах, другие позволяют изучить свойства самих систем, а именно: вид собственных функций линеаризаций упомянутых иерархий, описание элементарных преобразований Дарбу обоих иерархий, отыскание представлений построенных собственных функций и двойственных им в терминах определителей Фредгольма.

Ключевые слова: псевдодифференциальные операторы; сопряженный оператор; свободный член; иерархия n-КдФ; иерархия КП; строгая иерархия КП; уравнения Лакса

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Introduction

The integrable hierarchies that play a central role in this paper consist of collections of compatible Lax equations. This Lax form exists for many important nonlinear equations from mathematical physics. We illustrate it at the hand of the well-known example of the Korteweg – de Vries (KdV) equation that describes the propagation of shallow water waves in a narrow channel:

$$4u_t = 6uu_x + u_{xxx}, \tag{0.1}$$

where $u(x,t)$ corresponds to the height of the waves in the channel and depends on the space coordinate x along the channel and the time coordinate t. Let ∂ be the operator $\frac{\partial}{\partial x}$. The nonlinear partial differential equation (0.1) is then equivalent to the following identity between linear operators in ∂

$$\partial_t(\mathcal{L}_2) := \frac{\partial}{\partial t}(\mathcal{L}_2) = 0.\partial^2 + \partial_t(u) = [A, \mathcal{L}_2]. \tag{0.2}$$

Here $\mathcal{L}_2$ is the second order operator $\partial^2 + u$, also known as the *Schrödinger operator*, A is the third order operator $\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x$, and $[A, \mathcal{L}_2] := A\mathcal{L}_2 - \mathcal{L}_2 A$ is the commutator of $\mathcal{L}_2$ and A. Let R be some commutative algebra of functions in x and t that contains all the coefficients of $\mathcal{L}_2$, A and $[A, \mathcal{L}_2]$ for which ∂ and ∂_t are derivations of R. Then all three operators belong to the algebra $R[\partial]$ of differential operators in ∂ with coefficients from R and identity (0.2) becomes an equality of differential operators, where two elements of $R[\partial]$ are called equal if the coefficients of corresponding powers of ∂ are. The relation between the KdV-equation and the operator form (0.2) was first found by P. Lax (see [1]), which explains the name *Lax form* of the KdV equation for (0.2).

We focus here on two issues related to the form of equation (0.2). First of all, note the special character of this identity: it says namely that the commutator of a second order differential operator in ∂ and a third order one is a zero-th order operator in ∂, which is far from true in general. Moreover, the commutator on the right hand side of the identity suggests that $\mathcal{L}_2$ is a deformation of an operator independent of t by conjugation with an invertible time-dependent operator. For, if $\mathcal{L}_2 = KL_0K^{-1}$, with $\partial_t(L_0) = 0$, then there holds

$$\partial_t(\mathcal{L}_2) = \partial_t(K)L_0K^{-1} - KL_0K^{-1}\partial_t(K)K^{-1} = [\partial_t(K)K^{-1}, \mathcal{L}_2].$$

However, $R[\partial]$ does not seem to be the right framework to make sense out of this, for it lacks sufficient invertible elements. These considerations lead to a number of natural questions:

(a) Is there some algebraic relation linking the operators $\mathcal{L}_2$ and A or was it simply pure coincidence?

(b) Given $\mathcal{L}_2$, are there more differential operators $\widetilde{A}$ in $R[\partial]$ and derivations $\partial_{\widetilde{A}}$ of R such that

$$\partial_{\widetilde{A}}(\mathcal{L}_2) = [\widetilde{A}, \mathcal{L}_2]?$$

In particular, the commutator $[\widetilde{A}, \mathcal{L}_2]$ has then degree zero in ∂.

(c) Is there a framework, where the deformation picture $\mathcal{L}_2 = KL_0K^{-1}$ holds?

All three questions can be solved in an extension Psd of $R[\partial]$ called the *algebra of pseudo differential operators*. The structure of $R[\partial]$, its extension Psd and the properties that allow you to solve the questions (a), (b) and (c) are treated in Section 1. The next section treats the properties related with two decompositions of Psd that lead to the characterization of the wave functions of the two central hierarchies, their expressions in special functions and the elementary Darboux transformations with whom one can construct new solutions out of known ones.

1. The algebra of pseudo differential operators

In the introduction we already met the algebra $R[\partial]$. We start with giving precise conditions on R and ∂ under which we can form the extension Psd, keeping the perspective of application as wide as possible. So we let R be some commutative algebra over a field k of characteristic zero and let ∂ be a k-linear derivation defined on R. In this way one can choose to work in a real or complex context. Consider now the collection $R[\partial]$ of k-linear endomorphisms of R of the form $\sum_{i=0}^n a_i\partial^i, a_i \in R,$ i.e. the maps

$$R \ni r \mapsto \sum_{i=0}^{n} a_i\partial^i(r) \in R.$$

We call the elements of $R[\partial]$ *differential operators in* ∂ *with coefficients from* R. By the Leibniz property of ∂, the composition of "applying ∂^m" and "multiplying with $r_0 \in R$" is given by

$$\partial^m \circ r_0 = \sum_{i=0}^{m} \binom{m}{i} \partial^i(r_0)\partial^{m-i},$$

which belongs also to $R[\partial]$. Hence $R[\partial]$ is an algebra w.r.t. the composition of endomorphisms of R. In the rest of this paper we leave out this composition sign " $\circ$" and the multiplication between $\sum_i a_i\partial^i$ and $\sum_j b_j\partial^j$ in $R[\partial]$ is given by:

$$\sum_{i,j}\sum_{k=0}^{i} \binom{i}{k} a_i\partial^k(b_j)\partial^{j+i-k}. \tag{1.1}$$

As was mentioned in the introduction, we want to be able to translate identities between differential operators from $R[\partial]$ directly into identities for their coefficients. Therefore we will assume that all $\{\partial^i \mid i \geqslant 0\}$ are R-linear independent in $R[\partial]$. If this condition is not satisfied, then one has to pass to a cover of $R[\partial]$, where these relations are decoupled ([2]), and make the extension of the cover.

Under the assumption just mentioned, the algebra $R[\partial]$ can be extended to the algebra of pseudo differential operators by adding the inverses of all the powers of ∂ and by allowing infinite sums of these negative powers. One can view this process as adding "integral operators" to the differential operators of $R[\partial]$, but it is done in a purely algebraic way. First of all, one wants all the integral powers of ∂ to satisfy

$$\partial^n \partial^m = \partial^{n+m}, n, m \in \mathbb{Z}, \partial^0 = 1.$$

Next one uses the relation $\partial\, r = \partial(r) + r\partial,\ r \in R,$ and gets $\partial^{-1}\, r = r\partial^{-1} + \partial^{-1}\partial(r)\partial^{-1}$ or

$$\partial^{-1}\, r = r\partial^{-1} - \partial(r)\partial^{-2} + \partial^2(r)\partial^{-3} - \dots ,$$

and from the last formula, by induction,

$$\partial^{-n}\, r = \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} \partial^k(r)\partial^{-n-k},\ n > 0.$$

Thus one arrives at the set Psd$= R[\partial, \partial^{-1})$ of all formal series

$$P = \sum_{j=-\infty}^{N} p_j \partial^j, p_j \in R,$$

and letting for each $n \in \mathbb{Z}$,

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!},\ \text{if } k \geqslant 1,\ \text{and } \binom{n}{0} := 1, \tag{1.2}$$

it can be verified that the product of two series $P = \sum_i p_i \partial^i$ and $Q = \sum_j q_j \partial^j$ in Psd is defined by a formula that reminds (1.1), namely

$$PQ := \sum_i \sum_j \sum_{s=0}^{\infty} \binom{i}{s} p_i \partial^s(q_j)\partial^{j+i-s}. \tag{1.3}$$

In this way Psd becomes an associative algebra. A pseudo differential operator $P = \sum_{j=-\infty}^{N} p_j \partial^j$ is said to be *of order* N if $p_N \neq 0$. It is convenient to use at many computations in Psd the notation

$$P = \sum_{j=-\infty}^{N} p_j \partial^j = p_N \partial^N + \text{l.o.},$$

where l.o. is short for the lower order part $\sum_{j=-\infty}^{N-1} p_j \partial^j$ of P.

The algebra of pseudo differential operators admits a number of decompositions. For $s \in \mathbb{Z}$, any operator $P = \sum_j p_j \partial^j \in$ Psd can be split as

$$P = P_{<s} + P_{\geqslant s},\ \text{where } P_{<s} = \sum_{j<s} p_j \partial^j \text{ and } P_{\geqslant s} = \sum_{j \geqslant s} p_j \partial^j. \tag{1.4}$$

For $s = 0$, this yields in particular the splitting of P in the strict integral operator part $P_{<0}$ of P and its differential operator part $P_{\geqslant 0}$. Similarly, we have

$$P = P_{\leqslant s} + P_{>s},\ \text{where } P_{\leqslant s} = \sum_{j \leqslant s} p_j \partial^j \text{ and } P_{>s} = \sum_{j>s} p_j \partial^j. \tag{1.5}$$

For $s=0$, this corresponds to writing P as the sum of its integral operator part $P_{\leqslant 0}$ and its pure differential operator part $P_{>0}$.

Being an associative k-algebra, Psd is a Lie algebra over k with respect to the commutator. From the multiplication rules in Psd it follows that for $s=0$ the two decompositions (1.4) and (1.5) yield two ways to split the Lie algebra Psd into the direct sum of two Lie subalgebras. The first is given by

$$\mathrm{Psd} = \{P \in \mathrm{Psd}, P = P_{<0}\} \oplus \{P \in \mathrm{Psd}, P = P_{\geqslant 0}\} := \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geqslant 0}.$$

and the second one by

$$\mathrm{Psd} = \{P \in \mathrm{Psd}, P = P_{\leqslant 0}\} \oplus \{P \in \mathrm{Psd}, P = P_{>0}\} := \mathrm{Psd}_{\leqslant 0} \oplus \mathrm{Psd}_{>0}.$$

We denote by $\pi_{\geqslant 0}$ the projection from Psd on $\mathrm{Psd}_{\geqslant 0}$ consisting of taking the differential operator part of an element in Psd. Similarly, the projections of Psd on respectively $\mathrm{Psd}_{\leqslant 0}$, $\mathrm{Psd}_{>0}$, and $\mathrm{Psd}_{<0}$, we denote by respectively $\pi_{\leqslant 0}$, $\pi_{>0}$, and $\pi_{<0}$. Obviously, we have for every $P \in$ Psd

$$\pi_{\leqslant 0}(P) = P_{\leqslant 0}, \pi_{\geqslant 0}(P) = P_{\geqslant 0}, \pi_{>0}(P) = P_{>0}, \text{ and } \pi_{<0}(P) = P_{<0}.$$

A special role in our considerations is played by the *constant term* $\mathbf{ct}(P) := p_0$ of P.

Contrary to $R[\partial]$, the extended algebra Psd is rich in invertible elements. Let R^* denote the group of invertible elements in R.

Lemma 1.1. *Every pseudo differential operator* $P = \sum_{j \leqslant m} p_j \partial^j$ *with* $p_m \in R^*$ *has an inverse* P^{-1} *of the form* $\sum_{i \leqslant -m} q_i \partial^i$, *with* $q_{-m} = p_m^{-1}$.

P r o o f. The product of the elements $\sum_{j \leqslant m} p_j \partial^j$ and $\sum_{i \leqslant -m} q_i \partial^i$ is by definition equal to

$$\sum_{j \leqslant m} \sum_{i \leqslant -m} \sum_{s=0}^{\infty} \binom{j}{s} p_j \partial^s(q_i) \partial^{j+i-s}$$

This is an operator of order $\leqslant 0$ and, if it has to equal 1, then the leading coefficient q_{-m} has to be the inverse of p_m and for all $k \geqslant 1$ there has to hold

$$\sum_{\substack{i,j,s \\ i+j-s=-k}} \binom{j}{s} p_j \partial^s(q_i) = p_m q_{-m-k} + \sum_{\substack{i,j,s \\ i+j-s=-k \\ i>-m-k}} \binom{j}{s} p_j \partial^s(q_i) = 0.$$

Since p_m is invertible, one can solve from this q_{-m-k} assuming all the q_i, with index $i > -m-k$, are known. This shows the existence of the inverse of P. □

This Lemma leads to two groups inside Psd that play a role in the sequel.

Corollary 1.1. *The subsets* $D(0)$ *and* $D(1)$ *defined by*

$$D(0) = \{p_0 + \sum_{j<0} p_j \partial^j \mid p_0 \in R^*\} \text{ resp. } D(1) = \{1 + \sum_{j<0} p_j \partial^j \in \mathrm{Psd}\}$$

are groups w.r.t. the multiplication on Psd.

The monic elements of positive order in Psd possess still another property.

P r o p o s i t i o n 1.1. *Consider for* $N \geqslant 1$ *any monic element*

$$P = \partial^N + \sum_{i\geqslant 1} p_i \partial^{N-i} = \sum_{i\geqslant 0} p_i \partial^{N-i}$$

of order $N \geqslant 1$ *in Psd. Then there is a unique monic pseudo differential operator of order one*

$$P^{\frac{1}{N}} = \partial + \sum_{i=1}^{\infty} \ell_i \partial^{1-i} = \sum_{i=0}^{\infty} \ell_i \partial^{1-i},$$

satisfying $(P^{\frac{1}{N}})^N = P$. *The operator* $P^{\frac{1}{N}}$ *is called the* N*-th root of* P. *Moreover each coefficient* ℓ_i *of* $P^{\frac{1}{N}}$ *is a polynomial expression in the elements*

$$\{\partial^{l_j}(p_j) \mid l_j \geqslant 0, j \leqslant i\}.$$

P r o o f. We will show that one can find the coefficients $\{\ell_i\}$ in a unique recurrent way. Both p_0 and ℓ_0 are equal to 1. Next we compare the other coefficients in the identity

$$\Big(\sum_{i_1=0}^{\infty} \ell_{i_1} \partial^{1-i_1}\Big) \cdots \Big(\sum_{i_N=0}^{\infty} \ell_{i_N} \partial^{1-i_N}\Big) = \sum_{i\geqslant 0} p_i \partial^{N-i}. \tag{1.6}$$

From the multiplication rules in Psd one sees that the term with ∂^{N-1} in P can only be obtained by choosing in the N-fold product of the operator $P^{\frac{1}{N}}$ in (1.6), $N-1$ times the term ∂ and once the term ℓ_1 and disregarding the lower order terms. This gives you that ℓ_1 is determined uniquely: $\ell_1 = \frac{1}{N} p_1$. Assume now that all $\{\ell_j \mid j \leqslant k\}$ are uniquely determined and each ℓ_j is a polynomial expression in the $\{\partial^s(p_i) \mid s \geqslant 0, i \leqslant j\}$. Then a similar reasoning as for $k=1$ shows that $p_{k+1} - N\ell_{k+1}$ is a polynomial expression in the $\{\partial^r(\ell_j) \mid j \leqslant k, r \geqslant 0\}$. Since these $\{\ell_j\}$ were unique and possessed the required property, this concludes the proof of the proposition. □

Applying Proposition 1.1 to the operator $\mathcal{L}_2 = \partial^2 + u$ from the introduction, yields a first order operator $L = (\mathcal{L}_2)^{\frac{1}{2}}$ of the form

$$L = \partial + \sum_{j=1}^{\infty} \ell_{j+1} \partial^{-j}. \tag{1.7}$$

Note that operators of the form (1.7) can be obtained by conjugating ∂ with an element $K \in D(1)$. Under a mild condition, all L of the form (1.7) have this form.

Lemma 1.2. *If* $\partial : R \to R$ *is surjective, then any* $P = \partial + \sum_{i=1}^{\infty} p_{i+1} \partial^{-i}$ *can be obtained by dressing the operator* ∂ *by an element of* $D(1)$.

P r o o f. The proof consists of solving step by step the equation $PK = K\partial$ with a $K \in D(1)$. For, if $K = 1 + \sum_{j>0} k_j \partial^{-j}$, the right hand side is equal to $\partial + \sum_{j>0} k_j \partial^{1-j}$ and the left hand side equals

$$PK = \Big(\partial + \sum_{i=1}^{\infty} p_{i+1} \partial^{-i}\Big)\Big(1 + \sum_{j>0} k_j \partial^{-j}\Big) = \partial + \sum_{i>0} k_i \partial^{1-i} +$$

$$\sum_{i>0} \partial(k_i) \partial^{-i} + \sum_{i=1}^{\infty} p_{i+1} \partial^{-i} + \sum_{i\geqslant 1} \sum_{j\geqslant 1} \sum_{l\geqslant 0} p_{i+1} \binom{-i}{l} \partial^l(k_j) \partial^{-i-j-l}.$$

This shows that we have to choose K such that

$$\sum_{j\geqslant 1}\partial(k_j)\partial^{-j}+\sum_{i=1}^{\infty}p_{i+1}\partial^{-i}+\sum_{i\geqslant 1}\sum_{j\geqslant 1}\sum_{l\geqslant 0}p_{i+1}\binom{-i}{l}\partial^{l}(k_j)\partial^{-i-j-l}=0.$$

The coefficient of ∂^{-1} in the expression in the left hand side is equal to $\partial(k_1)+p_2$ and thanks to the assumption on ∂ one can find a k_1 such that this coefficient is zero. Assuming that one has found $\{k_1,\cdots,k_m\}$ for $m\geqslant 1$ such that the coefficients of all the $\partial^{-l}, l\leqslant m,$ are zero, then the next coefficient has the form

$$\partial(k_{m+1})+p_{m+2}+\text{ polynomial expression in } \partial^{l}(k_i) \text{ and the } p_{j+1}, i\leqslant m \text{ and } j\leqslant m$$

and one can choose k_{m+1} such that this equals zero. Thus the coefficients of K can be found inductively. □

Hence, if ∂ is surjective, then $\mathcal{L}_2=K\partial^2K^{-1}, K\in D(1),$ and the deformation picture from question (c) in the introduction holds. We can also comment now the questions (a) and (b) from the introduction. As for question (a), one verifies directly that $A=((\mathcal{L}_2)^{3/2})_{\geqslant 0}$, the differential operator part of $(\mathcal{L}_2)^{3/2}$. Since for all $s\geqslant 1$ the operator $(\mathcal{L}_2)^{s/2}$ commutes with $\mathcal{L}_2$, we see that

$$[((\mathcal{L}_2)^{s/2})_{\geqslant 0},\mathcal{L}_2]=[\mathcal{L}_2,((\mathcal{L}_2)^{s/2})_{<0}]$$

and the right hand side of this equality is of order zero or less in ∂. This shows that the $\left\{A_s=((\mathcal{L}_2)^{s/2})_{\geqslant 0}\right\}$ are good candidates to consider question (b) and they span all $\widetilde{A}\in R[\partial]$ with this property, for there holds

P r o p o s i t i o n 1.2. *Denote as above $(\mathcal{L}_2)^{1/2}$ by L. Let $\widetilde{A}$ be a differential operator in $R[\partial]$ of order r such that the commutator $[\widetilde{A},\mathcal{L}_2]$ is of order smaller or equal to zero. Then there are $a_i\in \mathrm{Ker}(\partial), 0\leqslant i\leqslant r,$ such that*

$$\widetilde{A}=\sum_{i=0}^{r}a_i(L^i)_{\geqslant 0}.$$

P r o o f. Note first of all that the operator L has the form $L=\partial+\sum_{j=1}^{\infty}\ell_{j+1}\partial^{-j}$. Therefore, we have for all $i\geqslant 1$ that $(L^i)_{\geqslant 0}=\partial^i+$ l.o. and there follows by induction on the order r of $\widetilde{A}$ that $\widetilde{A}=\sum_{i=0}^{r}a_i(L^i)_{\geqslant 0}$. What remains to be shown is that $a_i\in\mathrm{Ker}(\partial)$ and for that we use the fact that the order of $[\widetilde{A},\mathcal{L}_2]$ in ∂ is $-\ell$, with $0\leqslant\ell\leqslant\infty$. We claim now that the order of the commutator $[\widetilde{A},L]$ is $-1-\ell$. In particular, if $\widetilde{A}$ commutes with $\mathcal{L}_2$ then it also commutes with L. Suppose,

$$[\widetilde{A},L]=\alpha\partial^m+\text{ l.o. , with }\alpha\neq 0.$$

Then we have the formula

$$\begin{aligned}[\widetilde{A},\mathcal{L}_2]=[\widetilde{A},L^2]&=[\widetilde{A},L]L+L[\widetilde{A},L]\\&=(\alpha\partial^m+\text{ l.o.})(\partial+\text{ l.o.})+(\partial+\text{ l.o.})(\alpha\partial^m+\text{ l.o.})\\&=2\alpha\partial^{m+1}+\text{ l.o.}\end{aligned}$$

Hence the order of $[\widetilde{A}, \mathcal{L}_2]$ is $m+1$, which shows that $m = -1-\ell$. As $\ell \geqslant 0$, we see that the order of $[\widetilde{A}, L]$ in ∂ is smaller than zero. On the other hand we have

$$[\widetilde{A}, L] = [a_r\partial^r + \text{ l.o.}, \partial + \text{ l.o.}] = -\partial(a_r)\partial^r + \text{ l.o.}$$

and thus $\partial(a_r) = 0$. As a_r is in the kernel of ∂, it commutes with all elements of Psd and we have $[a_r(L^r)_{\geqslant 0}, \mathcal{L}_2] = a_r[(L^r)_{\geqslant 0}, \mathcal{L}_2]$ and thus $a_r(L^r)_{\geqslant 0}$ satisfies the same property as $\widetilde{A}$. This holds then also for the differential operator $\widetilde{A} - a_r(L^r)_{\geqslant 0}$ and continuing in this fashion gives you that all the a_i belong to $\text{Ker}(\partial)$. This finishes the proof of the proposition. $\square$

So, we have seen that it makes sense for each $s \geqslant 1$ to look for Schrödinger operators $\mathcal{L}_2$ and a k-linear derivation ∂_s of R, commuting with ∂ such that the Lax equation

$$\partial_s(\mathcal{L}_2) = [A_s, \mathcal{L}_2] \tag{1.8}$$

holds. It were Gelfand and Dickey ([3]) who realized that it made also perfect sense to consider similar Lax equations for an n-th order analogue $\mathcal{L}_n$ of $\mathcal{L}_2$. Simply replace $\mathcal{L}_2$ by $\mathcal{L}_n$ and A_s by $((\mathcal{L}_n)^{s/n})_{\geqslant 0}$, where this fractional power exists thanks to Proposition 1.1.

Having found the examples $((\mathcal{L}_2)^{s/2})_{\geqslant 0}$ the next step was to consider not only single Lax equations, but actually a whole chain of them corresponding to all these examples and to construct solutions for the whole system. Much work in this direction was initiated by Sato and his school, see e.g. [4] and [5]. Thereto one had to consider a set of derivations $\{\partial_s \mid s \geqslant 1\}$ of R, all commuting with ∂, and one looks for Schrödinger operators $\mathcal{L}_2$ that satisfy all the equations

$$\partial_s(\mathcal{L}_2) = [A_s, \mathcal{L}_2], s \geqslant 1. \tag{1.9}$$

This system of equations is the so-called *KdV hierarchy* and the equations (1.9) are the *Lax equations* of the hierarchy. Also for the higher order analogue $\mathcal{L}_n$ of the Schrödinger operator one can unite the corresponding Lax equations and that yields the n-KdV hierarchy. Note that the n-th root of a solution of the n-KdV hierarchy satisfies the same Lax equations as the solution itself. There holds namely

P r o p o s i t i o n 1.3. *Let $\mathcal{L}_n$ be a solution of the n-KdV hierarchy, L its n-th root and A_s be the projection $(L^s)_{\geqslant 0}$, $s \geqslant 1$. Then L has the form (1.7) and satisfies the Lax equations*

$$\partial_s(L) = [(L^s)_{\geqslant 0}, L], s \geqslant 1. \tag{1.10}$$

P r o o f. The form of L is a consequence of the construction of the n-th root in Proposition 1.1. Assume there is an s for which equation (1.10) does not hold, i.e.

$$\partial_s(L) - [(L^s)_{\geqslant 0}, L] = \beta\partial^m + \text{ l.o. }, \beta \neq 0.$$

Since ∂_s and taking the commutator with A_s are derivations of Psd, we get for its n-th power

$$\partial_s(\mathcal{L}_n) - [(L^s)_{\geqslant 0}, \mathcal{L}_n] = \partial_s(L^n) - [(L^s)_{\geqslant 0}, L^n] = \sum_{j=1}^{n} L^{j-1}(\partial_s(L) - [(L^s)_{\geqslant 0}, L])L^{n-j}$$

$$= \sum_{j=1}^{n}(\partial^{j-1} + \text{ l.o.})(\beta\partial^m + \text{ l.o.})(\partial^{n-j}) = n\beta\partial^{m+n-1} + \text{ l.o.} \neq 0$$

and this contradicts the fact that the left hand side of this equality is zero, because $\mathcal{L}_n$ is a solution of the n-KdV hierarchy. Thus all the Lax equations (1.10) have to hold for L. $\square$

The equations (1.10) for any L in Psd of the form (1.7) are called the Lax equations of the *KP (Kadomtsev – Petviashvili) hierarchy*, as they imply for the coefficient ℓ_2 of L the KP equation. Such an L is seen as a prototype of dressing the operator ∂ with an element from $D(1)$. Proposition 1.3 shows how all n-KdV hierarchies are contained in the KP hierarchy. In [6] it was shown how to construct solutions of the KP hierarchy starting from an infinite dimensional Grassmanian of a separable Hilbert space.

A further step [7] is to consider deformations M of ∂ by dressing it with the wider class of invertible operators K from $D(0)$ and by requiring that M should satisfy a similar set of Lax equations as in (1.10), but this time A_s should be replaced by the strict differential part of M^s. This deformation is called the *strict KP hierarchy*. Besides its Lax form, the strict KP hierarchy possesses still two other descriptions: the zero curvature form [7] and the bilinear form [8]. Both the KP hierarchy and its strict version have natural Cauchy problems associated with them and their solvability is discussed in [9]. There exists also a geometric construction of solutions of the strict KP hierarchy in the style of [6]. The manifold from which solutions of the strict KP hierarchy can be constructed is a fiber bundle over the Grassmanian mentioned above with the projective space of a separable Hilbert space as the generic fiber. Details can be found in [10]. Moreover, the solutions constructed in [10] can be expressed in Fredholm determinants. This is described in [11].

We discuss in the next section some properties relating the constant term of pseudo differential operators and the projections $\pi_{<0}$ and $\pi_{\geqslant 0}$ of Psd.

2. Properties related to $\pi_{<0}$, $\pi_{\geqslant 0}$ and ct

We start with a number of properties relating the constant term and the projection $\pi_{<0}$ of Psd on $\mathrm{Psd}_{<0}$.

P r o p o s i t i o n 2.1. *For arbitrary $f, g \in R$ and each $P = \sum_{i=-\infty}^{N} p_i\partial^i \in \mathrm{Psd}$ there hold the following relations in Psd:*

$$(fPg)_{<0} = fP_{<0}g, \tag{2.1}$$

$$(\partial P)_{<0} = \partial P_{<0} - \mathbf{ct}(P\partial), \tag{2.2}$$

$$(P\partial)_{<0} = P_{<0}\partial - \mathbf{ct}(P\partial), \tag{2.3}$$

$$(P\partial^{-1})_{<0} = P_{<0}\partial^{-1} + \mathbf{ct}(P)\partial^{-1}. \tag{2.4}$$

P r o o f. As for the first property, just note that for each differential operator $Q \in R[\partial]$ any product $fQg \in R[\partial], f, g \in R$. Hence we have

$$(fPg)_{<0} = (fP_{\geqslant 0}g)_{<0} + (fP_{<0}g)_{<0} = 0 + (fP_{<0}g)_{<0}.$$

To get property (2.2) we multiply P with ∂ from the left and substitute $P = \sum_i p_i\partial^i$. This yields

$$(\partial P)_{<0} = (\partial \sum_i p_i\partial^i)_{<0} = \sum_{i<0} \partial(p_i)\partial^i + \sum_{i+1<0} p_i\partial^{i+1}$$
$$= \sum_{i<0} \partial(p_i)\partial^i + \sum_{i<0} p_i\partial^{i+1} - p_{-1}\partial^0 = \partial P_{<0} - \mathbf{ct}(P\partial).$$

At the next two identities we proceed in a similar way:

$$(P\partial)_{<0} = (\sum_i p_i\partial^{i+1})_{<0} = \sum_{i+1<0} p_i\partial^{i+1}$$
$$= \sum_{i<0} p_i\partial^{i+1} - p_{-1} = P_{<0}\partial - \mathbf{ct}(P\partial)$$

and

$$(P\partial^{-1})_{<0} = (\sum_i p_i\partial^{i-1})_{<0} = \sum_{i-1<0} p_i\partial^{i-1}$$
$$= \sum_{i<0} p_i\partial^{i-1} + p_0\partial^{-1} = P_{<0}\partial^{-1} + \mathbf{ct}(P)\partial^{-1}.$$

This completes the proof of the proposition. □

The properties in Proposition 2.1 are used in the proof of the next proposition and at the characterization of the wave functions of the KP hierarchy and its strict version in [10].

On Psd there is important transformation of *taking the adjoint*. The *adjoint operator* P^* of $P = \sum_i p_i\partial^i$ is given by

$$P^* = \sum_i (-\partial)^i p_i = \sum_i (-1)^i \sum_{k=0}^{\infty} \binom{i}{k} \partial^k(p_i)\partial^{i-k}$$
$$= \sum_{\ell} \left\{ \sum_{k=0}^{\infty} (-1)^{\ell+k} \binom{\ell+k}{k} \partial^k(p_{\ell+k}) \right\} \partial^{\ell}. \tag{2.5}$$

Taking the adjoint, defined here quite formally, is an algebraic analogue of the following real analytic context. Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz–Bruhat functions on $\mathbb{R}$ with the inner product

$$< f \mid g >:= \int_{\mathbb{R}} f(x)g(x)dx.$$

Then considering on $\mathcal{S}(\mathbb{R})$ the linear operators of differentiating $\partial := \frac{d}{dx}$ and multiplying by a function $M_h : f \to hf, h \in \mathcal{S}(\mathbb{R}),$ one has for all $f, g \in \mathcal{S}(\mathbb{R})$,

$$< \partial f \mid g >=< f \mid -\partial g > \text{ and } < M_h f \mid g >=< f \mid M_h g >$$

which is often expressed as the adjoint of ∂ is $-\partial$ and the adjoint of M_h is M_h. So the adjoint of a linear operator on $\mathcal{S}(\mathbb{R})$ such as $D = \sum_{i=0}^{N} M_{h_i}\partial^i$ is equal to $\sum_{i=0}^{N}(-\partial)^i M_{h_i}$.

Lemma 2.1. *The operation of taking the adjoint on Psd is an anti-algebra morphism.*

P r o o f. Obviously, it is enough to check the equality $(P\,Q)^* = Q^*P^*$ for the operators $P = \partial^i$ and $Q = q,\ i \in \mathbb{Z},\ q \in R,$ i.e. one needs to prove that $(\partial^i\, q)^* = (-1)^i\, q\, \partial^i$. Using the multiplication rule (1.3) and the definition of the adjoint we get:

$$(\partial^i\, q)^* = \left(\sum_{s=0}^{\infty} \binom{i}{s} \partial^s(q)\partial^{i-s} \right)^* = \sum_{s=0}^{\infty} \binom{i}{s} (-1)^{i-s}\partial^{i-s}\partial^s(q)$$
$$= \sum_{s=0}^{\infty} \binom{i}{s} (-1)^{i-s} \left\{ \sum_{t=0}^{\infty} \binom{i-s}{t} \partial^{s+t}(q)\partial^{i-s-t} \right\}$$
$$= \sum_{k\leqslant i} \left\{ \sum_{t=0}^{\infty} (-1)^{k+t} \binom{i}{i-(k+t)} \binom{k+t}{t} \right\} \partial^{i-k}(q)\partial^k. \tag{2.6}$$

Now we calculate coefficients of ∂^k in (2.6). For $k = i$, we have

$$\sum_{t=0}^{\infty}(-1)^{i+t}\binom{i}{-t}\binom{i+t}{t}\partial^0(q) = (-1)^{i+0}\binom{i}{0}\binom{i}{0}q = (-1)^i\, q.$$

For $k < i$, taking into account relation (1.2), we get

$$\sum_{t=0}^{\infty}(-1)^{k+t}\binom{i}{i-(k+t)}\binom{k+t}{t} = \sum_{t=0}^{i-k}(-1)^{k+t}\binom{i}{(i-k)-t}\binom{k+t}{t} =$$
$$(-1)^k\binom{i}{i-k}\binom{k}{0} + (-1)^{k+1}\binom{i}{(i-k)-1}\binom{k+1}{1} + \ldots + (-1)^i\binom{i}{0}\binom{i}{i-k} =$$
$$(-1)^k\,\frac{i(i-1)\ldots(k+1)}{(i-k)!0!} + (-1)^{k+1}\,\frac{i(i-1)\ldots(k+1)}{((i-k)-1)!1!} + \ldots + (-1)^i\,\frac{i(i-1)\ldots(k+1)}{0!(i-k)!} =$$
$$(-1)^k\,\frac{i(i-1)\ldots(k+1)}{(i-k)!}\left\{\frac{(i-k)!}{(i-k)!0!} - \frac{(i-k)!}{((i-k)-1)!1!} + \ldots + (-1)^{i-k}\frac{(i-k)!}{0!(i-k)!}\right\}.$$

The expression in curly brackets here is the binomial formula for $(1-1)^{i-k}$, and hence is equal to zero. So we have $(\partial^i\, q)^* = (-1)^i\, q\partial^i$. □

By definition, the map $P \mapsto P^*$ maps $\mathrm{Psd}_{\geqslant 0}$ to $\mathrm{Psd}_{\geqslant 0}$ and $\mathrm{Psd}_{<0}$ to $\mathrm{Psd}_{<0}$ and since $P = (P^*)^*$, both maps are bijections. The adjoint plays a key role in the dual version of the hierarchies (see, e.g. [10]). Below we prove some relations between constant terms of certain combinations of pseudo differential operators and the projections $\pi_{\geqslant 0}$ and $\pi_{<0}$.

P r o p o s i t i o n 2.2. *For any* $f, g \in R$ *and each* $P = \sum_{i=-\infty}^{N} p_i\partial^i \in \mathrm{Psd}$ *there hold the formulae:*

$$\mathbf{ct}(Pf) = P_{\geqslant 0}(f), \tag{2.7}$$
$$\mathbf{ct}(\partial^{-1} fP\partial) = P^*_{\geqslant 0}(f), \tag{2.8}$$
$$(\partial^{-1}P)_{<0} = \partial^{-1}P_{<0} + \partial^{-1}\mathbf{ct}(P^*). \tag{2.9}$$

P r o o f. From the rule for multiplication in Psd follows that for each $f \in R$ and each $P \in \mathrm{Psd}$ the element $P_{<0}f$ belongs to $\mathrm{Psd}_{<0}$, hence we get that

$$\mathbf{ct}(Pf) = \mathbf{ct}(P_{\geqslant 0}f) = \mathbf{ct}\Big(\sum_{i=0}^{N} p_i\partial^i f\Big) = \sum_{i=0}^{N} p_i\partial^i(f) = P_{\geqslant 0}(f).$$

For the relation (2.8), we use first the multiplication rule (1.3) to compute the coefficients for $\partial^{-1} fP\partial$:

$$\partial^{-1} fP\partial = \partial^{-1}\sum_t p_t f\partial^{t+1} = \sum_i\sum_{s=0}^{\infty}\binom{-1}{s}\partial^s(p_{i-1}f)\partial^{i-s-1}.$$

Hence the constant term is given by

$$\sum_{i\geqslant 1}\binom{-1}{i-1}\partial^{i-1}(p_{i-1}f) = \sum_{t\geqslant 0}(-1)^t\partial^t(p_t f) =$$
$$\sum_{t\geqslant 0}\sum_{k=0}^{t}(-1)^t\binom{t}{k}\partial^{t-k}(p_t)\partial^k(f) = \sum_{k\geqslant 0}\left\{\sum_{t\geqslant 0}(-1)^t\binom{t}{k}\partial^{t-k}(p_t)\right\}\partial^k(f).$$

Now we compare with the expression for the coefficients of $P^*_{\geqslant 0} = \sum_{\ell\geqslant 0} p^*_\ell \partial^\ell$, namely

$$p^*_\ell = \sum_{r=0}^{\infty} (-1)^{\ell+r} \binom{\ell+r}{r} \partial^r(p_{\ell+r}) = \sum_{r=0}^{\infty} (-1)^{\ell+r} \binom{\ell+r}{\ell} \partial^r(p_{\ell+r}),$$

and we get the identity (2.8). Next we focus on relation (2.9). Since there clearly holds

$$(\partial^{-1}P)_{<0} = \partial^{-1}P_{<0} + \left(\partial^{-1}P_{\geqslant 0}\right)_{<0},$$

it suffices to show that $(\partial^{-1}P_{\geqslant 0})_{<0} = \partial^{-1}\mathbf{ct}(P^*)$. The constant term of P^* is by (2.5) given by

$$\mathbf{ct}(P^*) = \sum_{i=0}^{N} (-1)^i \partial^i(p_i).$$

Hence, this yields for the right hand side of the desired equality

$$\partial^{-1}\mathbf{ct}(P^*) = \sum_{s=0}^{\infty}\sum_{i=0}^{N} (-1)^s \partial^s((-1)^i\partial^i(p_i))\partial^{-1-s} = \sum_{s=0}^{\infty}\sum_{i=0}^{N} (-1)^{s+i}\partial^{s+i}(p_i)\partial^{-1-s}.$$

Next we compute the left hand side

$$(\partial^{-1}P_{\geqslant 0})_{<0} = \Big(\sum_{i=0}^{N}\sum_{r=0}^{\infty} (-1)^r\partial^r(p_i)\partial^{i-1-r}\Big)_{<0} = \sum_{i=0}^{N}\sum_{r\geqslant i} (-1)^r\partial^r(p_i)\partial^{i-1-r}$$
$$= \sum_{s=0}^{\infty}\sum_{i=0}^{N} (-1)^{s+i}\partial^{s+i}(p_i)\partial^{-1-s}.$$

This concludes the proof. □

Propositions 2.1 and 2.2 played a crucial role in [10] and [11] at the description of elementary Darboux transformations of both hierarchies and at expressing the constructed wave functions of the two hierarchies and their duals in Fredholm determinants.

References

[1] P. D. Lax, "Integrals of nonlinear equations of evolution and solitary waves", *Commun. Pure Appl. Math.*, **21**:5 (1968), 467–490.

[2] G. Wilson, "Commuting flows and conservation laws for Lax equations", *Math. Proc. Camb. Phil. Soc.*, **86**:1 (1979), 131–143.

[3] I. M. Gelfand, L. A. Dickey, "Fractional powers of operators and Hamiltonian systems", *Funct. Anal. Its Appl.*, **10**:4 (1976), 259–273.

[4] M. Sato, Y. Sato, "Soliton equations as dynamical systems on infinite-dimensional Grassman manifold", *Nonlinear Partial Differential Equations In Applied Science*, Proceedings of the U.S.-Japan seminar "Nonlinear partial differential equations in applied science" (Tokyo, 1982 (North-Holland mathematics studies)), 1983, 259–272.

[5] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, "Transformation groups for soliton equations", *Non-Linear Integrable Systems–Classical Theory and Quantum Theory*, Proceedings of RIMS symposium "Non-linear integrable systems–classical theory and quantum theory" (Kyoto, Japan, 13-16 May, 1981), 1983, 39–119.

[6] G. Segal, G. Wilson, "Loop groups and equations of KdV type", *Publications Mathematiques de l'IHES*, **61** (1985), 5–65.

[7] G. F. Helminck, A. G. Helminck, E. A. Panasenko, "Integrable deformations in the algebra of pseudo differential operators from a Lie algebraic perspective", *Theoret. and Math. Phys.*, **174**:1 (2013), 134–153.

[8] G. F. Helminck, E. A. Panasenko, S. V. Polenkova, "Bilinear equations for the strict KP hierarchy", *Theoret. and Math. Phys.*, **185**:3 (2015), 1804–1816.

[9] G. F. Helminck, A. G. Helminck, E. A. Panasenko, "Cauchy problems related to integrable deformations of pseudo differential operators", *Journal of Geometry and Physics*, **85** (2014), 196–205.

[10] G. F. Helminck, E. A. Panasenko, "Geometric solutions of the strict KP hierarchy", *Theoret. and Math. Phys.*, **198**:3 (2019), 48–68.

[11] G. F. Helminck, E. A. Panasenko, "Expressions in Fredholm determinants for solutions of the strict KP hierarchy", *Theoret. and Math. Phys.*, **199**:2 (2019), 637–651.

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