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COVERING MAPPINGS IN THE THEORY OF IMPLICIT SINGULAR DIFFERENTIAL EQUATIONS

© **A. I. Shindiapin** [1] **, E. S. Zhukovskiy** [2]

[1] Universidade Eduardo Mondlane
257 Praca 25 de Junho, Maputo, Mocambique, CP 257
E-mail: andrei.olga@tvcabo.co.mz
[2] Tambov State University named after G.R. Derzhavin
33 Internatsionalnaya St., Tambov, Russian Federation, 392000
The Peoples' Friendship University of Russia
6 Miklukho-Maklay St., Moscow, Russian Federation, 117198
E-mail: zukovskys@mail.ru

We propose method of studying implicit singular differential equations based on the results of the covering mapping theory. The article consists of three sections. In the first section we give the necessary designations and definitions and formulate the theorem on Lipcshitz perturbations of covering mappings. In the second section we introduce special spaces of measurable functions making it possible to study singular equations by methods of functional analysis and formulate the results about the Nemytskii operator in those spaces. In the last section we provide conditions for the resolubility of the Cauchy problem for implicit singular differential equations.
Key words: Implicit singular differential equation; Cauchy problem; covering mapping; Lipschitz mapping; metric space

Some problems of physics, aerodynamics, and technological processes can be reduced to singular differential equations with non-summable coefficients [1], [2]. The images of corresponding mappings, used to model such cases, do not belong to Lebesgue's spaces, which result in certain difficulties in studying such processes. The paper [3] proposed a space of summable functions, in which singular mappings become regular ones. This makes it possible to apply methods of classical analysis to differential equations with non-summable singularities. In studying the nonholonomous mechanical systems (see, for example, [4]) it is necessity to deal with the implicit singular equations. The standard method to study implicit differential equations is to use theorems on implicit functions. However, such theorems cannot be applied if the function generating the differential equation is not smooth enough or it has degenerated derivative. Additional difficulties appear in the case of singular equations. In literature we could not find methods to study such equations.

Recent rapid developments of the theory of covering mappings povides new opportunities in studying singular implicit differential equations. The authors in [5] proved a theorem on non-linear Lipcshitz perturbations of covering mappings and a proposed a new approach to study the implicit equations based on this theorem. Those studies have been further developed in [6], [7].

In this article, we build on [3], [5]–[7] and propose a formalization of implicit singular differential equation in a form of an equation with covering mapping in a special space of measurable functions. As a result, new conditions for existence of solutions of the Cauchy problem for such equations and its estimations have been found.

1. Covering mappings of metric spaces

We will use the following definitions.

Let (X,ρ_X), (Y,ρ_Y) be metric spaces. Denote the closed ball in space X with center in x and radius $r>0$ by $B_X(x,r)$.

D e f i n i t i o n 1 [8]. The mapping $\Psi:X\to Y$ is an α-*covering* (covering with constant $\alpha>0$), if for any $r>0$ and $u\in X$ there holds the inclusion

$$\Psi\big(B_X(u,r)\big)\supset B_Y\big(\Psi(u),\alpha r\big). \tag{1}$$

The mapping $\Psi:X\to Y$ is α-*covering* if and only if

$$\forall u\in X \;\; \forall y\in Y \;\; \exists x\in X \quad \Psi(x)=y, \;\; \rho_X(x,u)\leqslant \alpha^{-1}\rho_Y\big(y,\Psi(u)\big). \tag{2}$$

Note that the covering mapping is surjective. This follows from the inclusion (1), as any point $y\in Y$ belongs to the ball $B_Y\big(\Psi(u),\alpha r\big)$ for sufficiently big r. In the problems that follow, it is sufficient to consider the estimation (2) only for $y\in\Psi(X)$ and not for any $y\in Y$. This allows to ease the definition of covering mapping in our case.

D e f i n i t i o n 2 [5]. The mapping $\Psi:X\to Y$ is called *conditionally* α-*covering* (conditionally covering with constant $\alpha>0$), if for any $r>0$ and $u\in X$ there holds the

$$\Psi\big(B_X(u,r)\big)\supset B_Y\big(\Psi(u),\alpha r\big)\cap\Psi(X).$$

The mapping $\Psi:X\to Y$ is *conditionally* α-*covering* if and only if

$$\forall u\in X \;\; \forall y\in \Psi(X) \;\; \exists x\in X \quad \Psi(x)=y, \;\; \rho_X(x,u)\leqslant \alpha^{-1}\rho_Y\big(y,\Psi(u)\big). \tag{3}$$

The following proposition, proved in [8], is generalization of A.A.Milyuting's well-known theorem on Lipschit's perturbations of covering mappings. We will formulate this result for a particular case in order not to complicate reading with the unnecessary details, which do not effect our results.

Let $\Phi:X^2\to Y$ be a mapping and $y\in Y$. Consider equation

$$F(x)\doteq\Phi(x,x)=y. \tag{4}$$

T h e o r e m 1 [5]. *Let metric space X be complete and let the following conditions hold:*

- *for all $x\in X$ the mapping $\Phi(\cdot,x):X\to Y$ is conditionally α-covering, closed and*
$$y\in\Phi(X,x); \tag{5}$$
- *for all $x\in X$ the mapping $\Phi(x,\cdot):X\to Y$ is β-Lipschitz;*

Then, if $\beta<\alpha$, the equation (4) *has a solution and for any $u^0\in X$ then there exists solution $x=\xi\in X$ of this equation, which satisfy the inequality*

$$\rho_X(\xi,u^0)\leqslant\frac{1}{\alpha-\beta}\rho_Y\big(y,\Phi(u^0,u^0)\big). \tag{6}$$

N o t e 1. If the mapping $\Phi(\cdot,x):X\to Y$ is a covering one, then the inclusion (5) is trivial, and should be excluded from conditions of Theorem 1. In this case the equation (4) has a solution for any righthand part $y\in Y$. Moreover, as the solution satisfying the condition (6) exists, that means that the mapping $F:X\to Y$, $F(x)=\Phi(x,x)$ is $(\alpha-\beta)$-covering. Thus the theorem 1 gives the conditions of stability of the covering property in respect of Lipschitz perturbations.

2. Conditions of covering for the Nemytskii singular operator

Let $z_0, v : [a,b] \to \mathbb{R}$, $v(t) \geq 0$ for almost all $t \in [a,b]$ be two measurable functions. Following [3], we define the metric space $L_\infty([a,b],\mathbb{R},z_0,v)$ of all measurable functions $z:[a,b]\to\mathbb{R}$, for which the function $t\in[a,b]\mapsto \frac{z(t)-z_0(t)}{v(t)}$ is essentially bounded, with the distance

$$\rho_{L_\infty}(z_2,z_1) = \operatorname{vrai\,sup}_{t\in[a,b]} \frac{|z_2(t)-z_1(t)|}{v(t)}.$$

The metric space $L_\infty([a,b],\mathbb{R},z_0,v)$ is complete. In its definition we will drop the symbol z_0, if $z_0(t)\equiv 0$, and the symbol v, if $v(t)\equiv 1$. In particular, $L_\infty([a,b],\mathbb{R})$ is a "standard"Banach space of essentially bounded functions.

In order to apply the Theorem 1 to the implicit singular differential equations we will need covering conditions for the Nemytsky operator in the spaces $L_\infty([a,b],\mathbb{R},z_0,v)$. Let us formulate the corresponding proposition.

Let $z_0, v:[a,b]\to\mathbb{R}$ be measurable functions and the function $g:[a,b]\times\mathbb{R}\to\mathbb{R}$ satisfy the Caratheodory conditions, i.e. measurable on the first and continuous on the second variable. Let $y_0(t)=g\big(t,z_0(t)\big)$. We assume that for any $r>0$ there exists such $R>0$, such that for almost all $t\in[a,b]$ and for each $u\in\big[z_0(t)-rv(t),\ z_0(t)+rv(t)\big]$ we have $|g(t,u)-y_0(t)|\leq R$.

Let function g generate the Nemytskii operator

$$(N_g z)(t) \doteq g\big(t,z(t)\big), \quad t\in[a,b]. \tag{7}$$

Under the above conditions, this is a closed from $L_\infty([a,b],\mathbb{R},z_0,v)$ to $L_\infty([a,b],\mathbb{R},y_0)$.

Define $G_v:[a,b]\times\mathbb{R}\to\mathbb{R}$ as

$$G_v(t,u) \doteq g\big(t,\ u(t)v\big).$$

T h e o r e m 2. *Suppose there exists* $\alpha>0$, *such that for almost all* $t\in[a,b]$ *the mapping* $G_v(t,\cdot):\mathbb{R}\to\mathbb{R}$ *is an* α*-covering. Then Nemytskii operator* (7) *defined by* $N_g: L_\infty([a,b],\mathbb{R},z_0,v)\to$ $\to L_\infty([a,b],\mathbb{R},y_0)$ *is also an* α*-covering. If the mapping* $G_v(t,\cdot)$ *for almost all* $t\in[a,b]$ *is conditionally an* α*-covering, then the operator* (7) *is also conditionally an* α*-covering.*

Let us illustrate the use Theorem 2 to study the Nemytskii operator generated by the following simple function.

E x a m p l e 1. Let

$$g:[0,1]\times\mathbb{R}\to\mathbb{R}, \quad g(t,z)=t(1-t)z.$$

Let $v(t)=t(1-t)$, $z_0(t)\equiv 0$, $y_0(t)\equiv 0$. For such a function g the operator acts from the space $L_\infty([a,b],\mathbb{R},v)$ into space $L_\infty([a,b],\mathbb{R})$. As for any $t\in[0,1]$ function $G_v(t,u)=u$ is 1-covering by variable u, the operator $N_g: L_\infty([a,b],\mathbb{R},v)\to L_\infty([a,b],\mathbb{R})$ is also 1-covering.

3. The Cauchy problem for the implicit singular differential equation

Now we will use Theorems 1,2 to study the resolubility of the Cauchy problem for the implicit singular differential equation.

Suppose $z_0, v:[a,b]\to\mathbb{R}$ are summable functions, with $v(t)\geq 0$ for almost all $t\in[a,b]$. Suppose a summable function $f:[a,b]\times\mathbb{R}^2\to\mathbb{R}$, satisfies Caratheodory conditions, and $y:[a,b]\to\mathbb{R}$ is a measurable function and $\gamma\in\mathbb{R}$.

Consider the Cauchy problem

$$f\big(t,x(t),x'(t)\big)=y(t), \quad t\in[a,b], \quad x(a)=\gamma. \tag{8}$$

The solution of the Cauchy problem does not have to be defined on all the $[a,b]$, but must satisfy the given equation on $[a,c]$, for some $c\in(a,b]$. We will look for the solution of the Cauchy problem (8) in the class of $AC_\infty([a,c],\mathbb{R},z_0,v)$, absolutely continuous functions $x:[a,c]\to\mathbb{R}$ such that $x'\in L_\infty([a,c],\mathbb{R},z_0,v)$. We call the solution $x_{\widetilde{c}}\in AC_\infty([a,\widetilde{c}],\mathbb{R},z_0,v)$ the continuation of the solution $x_c\in AC_\infty([a,c],\mathbb{R},z_0,v)$, if $\widetilde{c}\geq c$ and $x_{\widetilde{c}}(t)=x_c(t)$ for $t\in[a,c]$. If there exists the continuation of the solution x_c then we call x_c continuable. We will call function $x:[a,d)\to\mathbb{R}$, maximally continued solution if the restriction of x onto $[a,c]$ for any $c\in(a,d)$ is a solution of (8) and

$$\lim_{c\to d-0}\ \operatorname*{vrai\,sup}_{t\in[a,c]}\frac{|x'(t)-z_0(t)|}{v(t)}=\infty.$$

We now give conditions that imply the resolubility of the Cauchy problem (8). We define function

$$y_0(t)=f\big(t,\gamma+\int_a^t z_0(s)\,ds,z_0(t)\big).$$

We assume that for any $r>0$ there exists $R>0$, such that for almost all $t\in[a,b]$ for every $x\in[-r,\,r]$, $u\in\big[z_0(t)-rv(t),\,z_0(t)+rv(t)\big]$ the inequality $|f(t,x,u)-y_0(t)|\leq R$ holds. Let $F_v:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ be defined as

$$F_v(t,x,u)\doteq f\big(t,\,x,\,u(t)v\big).$$

T h e o r e m 3. *Suppose function* $F_v(t,x,\cdot):\mathbb{R}\to\mathbb{R}$ *be conditionally covering for almost all* $t\in[a,b]$, *and for any* $x\in\mathbb{R}$ *and* $u\in\mathbb{R}$. *Suppose further the function* $F_v(t,\cdot,u):\mathbb{R}\to\mathbb{R}$ *is Lipschitz and* $y(t)\in f\big(t,x,\mathbb{R}\big)$. *Then there exists* $c\in(a,b]$ *and exists defined on* $[a,c]$ *solution* $x_c\in AC_\infty([a,c],\mathbb{R},z_0,v)$ *of the problem* (8). *Any solution of* (8) *is continuable to the solution defined on the whole* $[a,b]$ *or to the maximally continued solution.*

P r o o f of this theorem is based on the representation of the problem (8) in a form of operator equation (4) in respect of the derivative of the unknown function. In our case the mapping $\Phi(\cdot,u):L_\infty([a,c],\mathbb{R},z_0,v)\to L_\infty([a,c],\mathbb{R},y_0)$ is a Nemytskii operator, which due to Theorem 2 is conditionally an α-covering; and the mapping $\Phi(u,\cdot):L_\infty([a,c],\mathbb{R},z_0,v)\to L_\infty([a,c],\mathbb{R},y_0)$ is an integral operator, which is Lipschitz with the constant β, which is less than α when c is close to a. Thus, by Theorem ,1 problem (8) has a solution. The continuation of each solution is proved in a similar way.

REFERENCES

1. *Beloretskiy S.M., Lifanov I.K., Mihailov A.A.* Computer modeling of the broken current of profiles with angle points // Proceedings of the USSR Academy of Sciences. 1985. V. 285. № 3. P. 621–629.

2. *Kiguradze I.T., Shehter B.L.* Singular boundary value problems for ordinary differential equations // Itogi nuki i tehniki. Ser. Contemporary broblems of mathematics: new achievements. V. 30. P. 105–201.

3. *Shindiapin A.I.* On linear singular functional-differential equations in one functional space // Abstract and Applied Analysis. 2004. № 7. P. 567–575.

4. *Zakalyukin I.V.* Particularities of equations of dynamics of some nonholonomous systems and implicit differential equations. M., 2010.

5. *Avakov E.R, Arutyunov A.V., Zhukovskiy E.S.* Covering mappings and its applications to the differential equations not solved in respect to derivative // Differential Equations. 2009. V. 45. № 5. P. 613–634.

6. *Arutyunov A.V., Zhukovskiy E.S. Zhukovskiy S.E.* On the correctness of the differential equations not solved in respect to derivative // Differential Equations. 2011. V. 47. № 11. P. 1523–1537.

7. *Zhukovskiy E.S., Pluzhnikova E.A.* Covering mappings in the product of metric spaces and boundary value problems for the differential equations not solved in respect to derivative. // Differential Equations. 2013. V. 49. № 4. P. 439–455.

8. *Arutyunov A.V.* Covering mappings in metric spaces and fixed points // Proceedings of Academy of Sciences. 2007. V. 416. № 2. P. 151–155.

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Shindiapin Andrey Igorevich, Eduardo Mondlane University, Mozambique, Doctor of Physics and Mathematics, Professor, Professor of the Department of Mathematics and Computer Science, e-mail: andrei.olga@tvcabo.co.mz

Zhukovskiy Evgeny Semenovich, Tambov State University named after G.R. Derzhavin, Tambov, the Russian Federation, Doctor of Physics and Mathematics, Professor, Director of the Research Institute of Mathematics, Physics and Informatics; Peoples' Friendship University of Russia, Moscow, the Russian Federation, Doctor of Physics and Mathematics, Professor of the Department of Nonlinear Analysis and Optimization, e-mail: zukovskys@mail.ru

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НАКРЫВАЮЩИЕ ОТОБРАЖЕНИЯ В ТЕОРИИ НЕЯВНЫХ СИНГУЛЯРНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

© **А. И. Шиндяпин [1], Е. С. Жуковский [2]**

[1] Университет имени Эдуардо Мондлане
CP 257 Мозамбик, Мапуто, Площадь 25 июня, 257
E-mail: andrei.olga@tvcabo.co.mz
[2] Тамбовский государственный университет им. Г.Р. Державина
392000, Российская Федерация, г. Тамбов, ул. Интернациональная, 33
Российский университет дружбы народов
117198, Российская Федерация, г. Москва, ул. Миклухо-Маклая, 6
E-mail: zukovskys@mail.ru

В работе предлагаются методы исследования неявных сингулярных дифференциальных уравнений, основанные на результатах о накрывающих отображениях. Статья состоит из трех параграфов. В первом параграфе приведены необходимые обозначения, определения, сформулирована теорема о липшицевых возмущениях накрывающих отображений; во втором — введены специальные метрические пространства измеримых функций, позволяющие применить к сингулярным уравнениям методы анализа, здесь получено утверждение о накрывающих свойствах оператора Немыцкого в таких пространствах; в третьем параграфе на основе перечисленных результатов получены условия разрешимости задачи Коши для неявного сингулярного дифференциального уравнения.

Ключевые слова: неявное сингулярное дифференциальное уравнение, задача Коши, накрывающее отображение, липшицево отображение, метрическое пространство

СПИСОК ЛИТЕРАТУРЫ

1. Белоцерковский С.М., Лифанов И.К., Михайлов А.А. Моделирование на ЭВМ отрывного обтекания профилей с угловыми точками // Докл. АН СССР. 1985. Т. 285. № 3. С. 621–629.

2. *Кигурадзе И.Т., Шехтер Б.Л.* Сингулярные краевые задачи для обыкновенных дифференциальных уравнений // Итоги науки и техники. Сер. Современные проблемы математики: новые достижения. 1987. Т. 30. С. 105–201.

3. *Шиндяпин А.И.* О краевой задаче для одного сингулярного уравнения // Дифференциальные уравнения. 1984. Т. 20. № 3. С. 450–455.

4. *Закалюкин И.В.* Особенности уравнений динамики некоторых неголономных систем и неявные дифференциальные уравнения // Автореферат дисс. канд. физ.-мат. наук: 01.02.01 [Место защиты: Московский авиационный институт (государственный технический университет)]. М., 2010.

5. *Аваков Е.Р., Арутюнов А.В., Жуковский Е.С.* Накрывающие отображения и их приложения к дифференциальным уравнениям, не разрешенным относительно производной // Дифференциальные уравнения. 2009. Т. 45. № 5. С. 613–634.

6. *Арутюнов А.В., Жуковский Е.С., Жуковский С.Е.* О корректности дифференциальных уравнений, не разрешенных относительно производной // Дифференциальные уравнения. 2011. Т. 47. № 11. С. 1523–1537.

7. *Жуковский Е.С., Плужникова Е.А.* Накрывающие отображения в произведении метрических пространств и краевые задачи для дифференциальных уравнений, не разрешенных относительно производной // Дифференциальные уравнения. 2013. Т. 49. № 4. С. 439–455.

8. *Арутюнов А.В.* Накрывающие отображения в метрических пространствах и неподвижные точки // Доклады Академии наук. 2007. Т. 416. № 2. С. 151–155.

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Шиндяпин Андрей Игоревич, Университет имени Эдуардо Мондлане, г. Мапуту, Мозамбик, доктор физико-математических наук, профессор, профессор кафедры математики и информатики, e-mail: andrei.olga@tvcabo.co.mz

Жуковский Евгений Семенович, Тамбовский государственный университет им. Г.Р. Державина, г. Тамбов, Российская Федерация, доктор физико-математических наук, профессор, директор научно-исследовательского института математики, физики и информатики; Российский университет дружбы народов, г. Москва, Российская Федерация, доктор физико-математических наук, профессор, профессор кафедры нелинейного анализа и оптимизации, e-mail: zukovskys@mail.ru

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