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THE LAPLACE TRANSFORM METHOD IN AN ALGORITHM OF SOLVING DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENT

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Key words: linear differential equations; delayed argument; symbolic-numerical algorithm.
The method is used for linear differential equations with delayed argument. There is constructed an algorithm, which is symbolic-numerical. The numerical component concerns a representation of functions, involved into the process by some kind of series.

1. Introduction

There is a class of physical problems, which is associated with action of some kind of complementary forces - forces which are involved at various not initial time moments. Such problems frequently lead to the so called differential equations with delayed argument. Different ways of dealing with such equations exist. See for example [1–4]. We consider linear equations with constant coefficients and right-hand parts of exponential increase.

Applications of the Laplace transform method are well known. In this article we continue working-out the application of Laplace transform for solving differential equations (for example [5–7]).

It permits to reduce an infinitesimal problem to an algebraic one that may be solved symbolically or symbolic-numerically. Moreover, it gives means to estimate an accuracy of calculations.

However there are some facts which prevent using this method in a symbolic way. Some difficulties, for example, are connected with a form of the solution of the Laplace image of the input differential equations, i.e. the exponential polynomials, which appears in the solution of algebraic equation. We suggest the usage of series expansion of some kind for symbolic-numerical solution with a necessary accuracy. It extends the class of equations to be solved by this method.

We restrict ourselves to the consideration of one equation, but the method works similarly with systems of equations of such type.

2. A differential equation with delayed argument

We consider all functions, either unknown or standing at the right-hand parts of equations, on the segment $\mathbf{T}: 0 \leqslant t \leqslant T$. Split $\mathbf{T}$ into parts by rational points $0 < t_k < t_{k+1} < T, k = 0, \dots, N$. All functions of the argument t are supposed to satisfy the conditions for existing of their Laplace transform, i.e. they have an exponential increase.

Consider the equation

$$x^{(n)}(t) + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} x^{(n-j)}(t - t_k) = f(t), \tag{1}$$

with initial conditions $x^{(n-j)}(0) = x_0^{(n-j)}, j = 1, \ldots, n$. As the right-hand members of equations we consider here a composite function $f(t)$, whose components are represented as finite sums of exponents with polynomial coefficients.

$$f(t) = f_k(t), \ t_k < t < t_{k+1}, \ k = 1, \ldots, N, \tag{2}$$

where

$$f_k(t) = \sum_{s_k=1}^{S_k} P_{s_k}(t) e^{b_{s_k} t}, \quad k = 1, \ldots, N, \tag{3}$$

and $P_{s_k}(t) = \sum_{m=0}^{M_{s_k}} c_{s_k m} t^m$.

3. Preparation for Laplace transform

The first step to prepare the equation (1) for symbolic performance of Laplace transform is presentation it by means of the Heaviside function $\eta(t)$. We obtain the following form of the equation (1):

$$x^{(n)}(t) + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} \eta(t - t_k) x^{(n-j)}(t - t_k) = f(t), \tag{4}$$

$f(t)$ must also be written by means of Heaviside function.

Represent $f(t)$ using the Heaviside function. At first we write

$$f(t) = \sum_{k=2}^{N-1} [f_k(t) - f_{k-1}(t)] \eta(t - t^k) + f_k(t) \eta(t).$$

Then transform $f_k(t) - f_{k-1}(t)$ into the function of $t - t_l^i$:

$$f_k(t) - f_{k-1}(t) = \phi_k(t - t_k).$$

The function $\phi_k(t - t_k)$ is represented as a finite sum

$$\phi_k(t - t_k) = \sum_{s=1}^{S_k} \psi_s^k(t - t_k) e^{b_s^k t_k} e^{b_s^k (t - tk)} - \sum_{s=1}^{S_{k-1}} \psi_s^{k-1}(t - t_k) e^{b_s^{k-1} t_k} e^{b_s^{k-1}(t - t_k)}.$$

Here $\psi_s^k(t - t_k) = P_s^k(t)$ and $\psi_s^k(t - t_k) = \sum_{m=0}^{M_s} \gamma_{sm}^k (t - t_k)^m$. Coefficients γ_{sm}^k are calculated by the formula

$$\gamma_{sm}^k = \sum_{j=0}^{M_s^l - m} c_{s,m+j}^l \binom{m+j}{j} (t^k)^j.$$

Finally the function $f(t)$ is reduced to the form

$$f(t) = \sum_{k=2}^{N-1} \phi^k(t - t_k) \eta(t - t_k) + \sum_{s=1}^{S_N} P_s^N(t) e^{b_{S_N}^1 t} \eta(t).$$

4. Laplace transform

It permits to write symbolically the Laplace image of the equation (1):

$$\left(p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j}\right)X(p)=\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p), \quad (5)$$

where $X(p)$ and $F(p)$ are the Laplace images of $x(t)$ and $f(t)$, correspondingly, and $F(p)$ is also a sum of exponents with polynomial coefficients.

The Laplace transform of $\phi_k(t-t_k)\eta(t-t_k)$ equals to

$$\Phi_k(p)=\left[\sum_{s=1}^{S_k}\sum_{m=0}^{M_s}\gamma_{ksm}e^{b_s^k t_k}\frac{m!}{(p-b_s^k)^{m+1}}-\sum_{s=1}^{S^{k-1}k}\sum_{m=0}^{M_s^{k-1}}\gamma_{sm}^{k,\,k-1}e^{b_s^{k-1}t_k}\frac{m!}{(p-b_s^{k-1})^{m+1}}\right]e^{-t_k p}.$$

Finally, the Laplace transform of $f(t)$ is the following:

$$F(p)=\sum_{k=2}^{N-1}\Phi^k(p)+\sum_{s=1}^{S}\sum_{m=0}^{M_s^1}c_{sm}\frac{m!}{(p-b_s^1)^{m+1}}. \quad (6)$$

5. Solving the algebraic equation

Denote

$$Q(p)=\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p),$$

$$D(p)=p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j},$$

then

$$X(p)=\frac{Q(p)}{D(p)}. \quad (7)$$

6. Inverse Laplace transform

6.1. Calculation of a half-plane of holomorphy

Consider $X(p)$ (7) and its denominator $D(p)$. There exists a half-plane, where $X(p)$ is holomorphic. To find it we must find a half-plane, where $D(p)$ is non-zero. Let us find $\sigma>0$ such that $D(p)\neq 0$ for all $p: Re p>\sigma$. As $D(p)\to\infty$ while $p\to\infty$ then for each $\delta>0$ there exists σ such that $D(p)>\delta$ if $p: Re p>\sigma$.

We have for sufficiently large $|p|$

$$|D(p)|\geqslant |p|^N(1-\sum_{j=1}^{n}\sum_{k=0}^{N}|a_{jk}||p|^{(n-j)/N}).$$

Denote $A=\sum_{j=1}^{n}\sum_{k=0}^{N}|a_{jk}|$, and take

$$\sigma=max\left\{\delta,\frac{\delta}{1-A}\right\}.$$

Then if $Re p>\sigma$, then

$$|D(p)|>\delta.$$

So we may take the half-plane $Rep > \sigma$, $X(p)$ is holomorphic in it.

We must mention, that the line $Rep = \tilde{\sigma}, \tilde{\sigma} \geqslant \sigma$, may be taken as line of integration for numerical calculation of the inverse Laplace transform.

6.2. Expansion of the solution in a series

Writing t_k as $t_k = \frac{\tau_k}{\sigma_k}$, denote $\sigma = LCM_k(\sigma_k)$, and $t_k = \frac{\tilde{\tau}_k}{\sigma}$.

Denote $e^{-\frac{p}{\sigma}} = z$. Then

$$X(p) = \frac{\sum_{j=1}^{n} p^{j-1} x_0^{(n-j)} + \sum_{j=1}^{n-1} \sum_{k=0}^{N} a_{jk} p^{j-1} x_0^{(n-j)} z^{\tilde{\tau}_k} + F(p)}{p^n + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} z^{\tilde{\tau}_k} p^{n-j}}. \tag{8}$$

We do not write the exact expression of such kind for $F(p)$, as it is rather bulky, mention only, that the exponents are the same, because we take the same split points. Formally we expand (8) in a Taylor serious by z at the point $z = 0$. It corresponds to $p : \mathrm{Re}p = +\infty$. Substituting $e^{-\frac{p}{\sigma}}$ instead of z , we obtain the series for $X(p)$ by $e^{-\frac{np}{\sigma}}$, which converges in some neighbourhood of ∞ :

$$\sum_{n} A_n e^{-\frac{np}{\sigma}}, \tag{9}$$

where A_n are proper fractions, and can be represented as sums of partial fractions.

6.3. Inverse Laplace transform

For the series (9) the Inverse Laplace transform may be written symbolically.

A problem is to define n and $\mathrm{Re}p$ sufficient for designed accuracy of the differential equation.

7. On accuracy

Let us take the $n-th$ Taylor approximation of $X(p)$ and find its inverse Laplace image. Denote by $\tilde{x}(t)$ an approximate solution of (1), which is equal to this image. The accuracy of such solution we denote by ε , i.e.

$$\max_{\mathbf{T}} |x(t) - \tilde{x}(t)| < \varepsilon. \tag{10}$$

The remainder term of (9) may be written in the form

$$\sum_{k=n} \frac{\alpha_k}{p^k} e^{-\frac{kp}{\sigma}}.$$

Demand

$$|p| \frac{|\alpha_n|}{(\mathrm{Re}p)^n} e^{-\frac{n\mathrm{Re}p}{\sigma}} < \varepsilon.$$

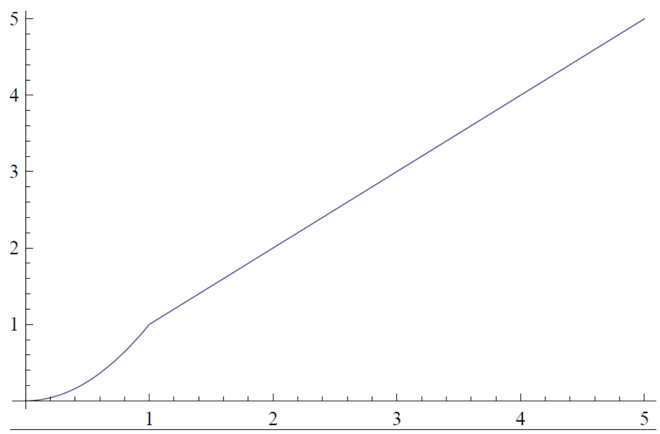
Then we obtain (10) for each $t \in \mathbf{T}$.

8. Example

Consider the equation

$$x'' + 2\eta(t-1)x'(t-1) - \eta(t-2)x'(t-2) + \eta(t-3)x'(t-3) + \eta(t-3)x(t-3) = f(t),$$

where $f(t) = t^2(\eta(t) - \eta(t-1)) + t\eta(t-1)$.



The Laplace image of the equation:

$$\left(p^2 + 2pe^{-p} - pe^{-2p} + pe^{-3p} + e^{-3p}\right) X = px_0 + x_0^1 + 2e^{-p}x_0 - e^{-2p}x_0 + e^{-3p}x_0 + F, \qquad (11)$$

where $F = 2/p^3 - (e^{-p}(2+p))/p^3$.

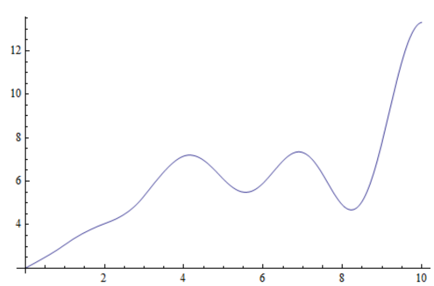
The solution of (9) is the following

$$X(p) = \frac{-2e^{-p} + 2 - e^{-p}p + 2p^3e^{-3p} - 2e^{-2p}p^3 + 4e^{-p}p^3 + p^3 + 2p^4}{p^3(e^{-3p} + pe^{-3p} - e^{-2p}p + 2e^{-p}p + p^2)}.$$

Expanding $X(p)$ into the exponential series described before and taking 10 terms of this series, we obtain

$$\begin{aligned} YP = {} & \frac{e^{-p}(-4 - 2p - p^2 - 2p^3)}{p^6} + \frac{e^{-2p}(8 + 6p + 2p^2 + 4p^3 + p^4)}{p^7} + \frac{2 + p^3 + 2p^4}{p^5} + \\ & \frac{e^{-3p}(-16 - 18p - 8p^2 - 9p^3 - 5p^4 - 3p^5)}{p^8} + \frac{e^{-4p}(32 + 48p + 28p^2 + 23p^3 + 17p^4 + 9p^5)}{p^9} + \\ & \frac{e^{-5p}(-64 - 120p - 88p^2 - 62p^3 - 49p^4 - 28p^5 - 4p^6)}{p^{10}} + \\ & \frac{e^{-7p}(-256 - 672p - 716p^2 - 520p^3 - 389p^4 - 263p^5 - 104p^6 - 19p^7)}{p^{12}} + \\ & \frac{e^{-6p}(128 + 288p + 258p^2 + 178p^3 + 138p^4 + 87p^5 + 25p^6 + 3p^7)}{p^{11}} + \\ & \frac{e^{-8p}(512 + 1536p + 1904p^2 + 1506p^3 + 1106p^4 + 775p^5 + 372p^6 + 95p^7 + 7p^8)}{p^{13}} + \\ & \frac{1}{p^{14}} e^{-9p}\left(-1024 - 3456p - 4896p^2 - 4274p^3 - 3168p^4 - 2255p^5 - 1232p^6 - 406p^7 - 61p^8 - 3p^9\right) + \\ & \frac{1}{p^3} e^{-10p}\left(2\left(-\frac{128}{p^9} - \frac{272}{p^8} - \frac{214}{p^7} - \frac{71}{p^6} - \frac{9}{p^5}\right)p^3 - 2\left(\frac{256}{p^{10}} + \frac{640}{p^9} + \frac{616}{p^8} + \frac{275}{p^7} + \frac{55}{p^6} + \frac{3}{p^5}\right)p^3 + \right. \\ & \quad \left(-\frac{512}{p^{11}} - \frac{1472}{p^{10}} - \frac{1680}{p^9} - \frac{945}{p^8} - \frac{265}{p^7} - \frac{31}{p^6} - \frac{1}{p^5}\right)\left(-2 - p + 4p^3\right) + \\ & \quad \left.\left(\frac{1024}{p^{12}} + \frac{3328}{p^{11}} + \frac{4400}{p^{10}} + \frac{2992}{p^9} + \frac{1090}{p^8} + \frac{197}{p^7} + \frac{14}{p^6}\right)\left(2 + p^3 + 2p^4\right)\right); \end{aligned}$$

The Inverse Laplace transform of this 10-th approximation is rather bulky, and we do not demonstrate it as a formula, but we present it the graphic form.



9. Conclusion

In the conclusion let us mention the advantages of our method:

1. The algebraization of the problem makes possible to apply fast and efficient method for solving algebraic linear system with polynomial coefficients. It is actual because it permits to solve huge problems.

2. The expansion into the series of exponent with polynomial coefficients extends the class of equations which may be solved by means of Laplace transform.

REFERENCES

1. *Asl F.M., Ulsoy A.G.* Analysis of a system of linear delay differential equations // Journal of Dynamic Systems, Measurement and Control. 2003. № 125. P. 215–223.
2. *Baker C.T.H., Paul C.A.H., Wille D.R.* Issues in the numerical solutions of evolutionary delay differential equations // Adv. Comp. Math. 1995. № 3. P. 171–196.
3. *Kitamura Y., Kusano T.* Vanishing oscillations of solutions of a class of differential systems with retarded argument // Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 1977. V. 62. P. 325–334.
4. *Kitamura Y., Kusano T.* Oscillation and a class of nonlinear differential systems with general deviating arguments // Nonlinear Anal. Theory Methods Appl. 1978. V. 2. P. 537–551.
5. *Malaschonok N.A.* An Algorithm for Symbolic Solving of Differential Equations and Estimation of Accuracy // Computer Algebra in Scientific Computing. LNCS 5743. Berlin: Springer, 2009. P. 213-225.
6. *Malashonok N.A., Rybakov M.A.* Symbolic-Numerical Solution of Systems of Linear Ordinary Differential Equations with Required Accuracy // Programming and Computer Software, Springer-Verlag, 2013. V. 39. № 3. P. 150-157.
7. *Malashonok N.* Symbolic solving of partial differential equation systems and compatibility conditions // Serdica J.Computing. 2013. V. 7. № 3. P. 199-214.

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Малашонок Н.А. ПРЕОБРАЗОВАНИЕ ЛАПЛАСА В АЛГОРИТМЕ РЕШЕНИЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

Представлен символьно-численный алгоритм решения линейных дифференциальных уравнений с запаздывающим аргументом. Численная компонента алгоритма состоит в представлении участвующих функций некоторыми функциональными рядами и оценке точности приближенного решения.

Ключевые слова: преобразование Лапласа; дифференциальные уравнения с запаздывающим аргументом; символьно-численный алгоритм.

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